5. Let $f(x) = x^2$. f is a polynomial function, so it is continuous and thus the intermediate value theorem applies. $f(1) = 1$ and $f(2) = 4$, so there is a number c between 1 and 2 such that $f(c) = 3$. By the definition of square root, $c = \sqrt{3}$, Q.E.D.

6. Prove that if f is continuous, and if $f(a)$ is positive and $f(b)$ is negative, then there is at least one zero of $f(x)$ between $x = a$ and $x = b$.

Proof:

f is continuous, so the intermediate value theorem applies. $f(a)$ is positive and $f(b)$ is negative, so there is a number $x = c$ between a and b for which $f(c) = 0$. Therefore, f has at least one zero between $x = a$ and $x = b$, Q.E.D.

7. The intermediate value theorem is called an existence theorem because it tells you that a number such as $\sqrt{3}$ *exists*. It does not tell you how to *calculate* that number.

8. Telephone your sweetheart's house. An answer to the call tells you the "existence" of the sweetheart at home. The call doesn't tell such things as how to get there, and so on. Also, getting no answer does not necessarily mean that your sweetheart is out.

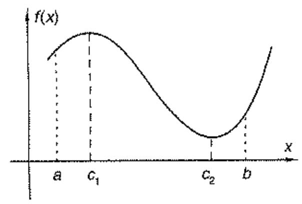
9. Let $f(t)$ = Jesse's speed – Kay's speed. $f(1) = 20 - 15 = 5$, which is *positive*. $f(3) = 17 - 19 = -2$, which is *negative*. The speeds are assumed to be continuous (because of laws of physics), so f is also continuous and the intermediate value theorem applies.
So there is a value of t between 1 and 3 for which $f(t) = 0$, meaning that Jesse and Kay are going at exactly the same speed at that time. The *existence* of the time tells you neither what that time is nor what the speed is. An existence theorem, such as the intermediate value theorem, does not tell these things.

10. Let $f(x)$ = number of dollars for x-ounce letter. f does *not* meet the hypothesis of the IVT on the interval [1, 9] because there is a step discontinuity at each integer value of x. There is no value of c for which $f(c) = 2$ because $f(x)$ jumps from 1.98 to 2.21 at $x = 8$.

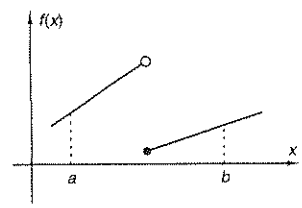
11. You must assume that the cosine is function continuous. Techniques:
   - $c = \cos^{-1} 0.6 = 0.9272...$
   - Using the solver feature, $c = 0.9272...$
   - Using the intersect feature, $c = 0.9272...$

12. You must assume that 2^x is continuous. $f(0) = 2^0 = 1$, because any positive number to the 0 power equals 1.
   - $c = \log_2 3 = \dfrac{\log 3}{\log 2} = 1.5849...$
   - Using the solver feature, $c = 1.5849...$
   - Using the intersect feature, $c = 1.5849...$

13. This means that a function graph has a high point and a low point on any interval in which the function is continuous.



If the function is *not* continuous, there may be a point missing where the maximum or minimum would have been.



Another possibility would be a graph with a vertical asymptote somewhere between a and b.

14. Prove that if f is continuous on $[a, b]$, the image of $[a, b]$ under f is all real numbers between the minimum and maximum values of $f(x)$, inclusive.

Proof:

By the extreme value theorem, there are numbers x_1 and x_2 in $[a, b]$ such that $f(x_1)$ and $f(x_2)$ are the minimum and maximum values of $f(x)$ on $[a, b]$. Because x_1 and x_2 are in $[a, b]$, f is continuous on the interval whose endpoints are x_1 and x_2. Thus, the intermediate value theorem applies on the latter interval. Thus, for any number y between $f(x_1)$ and $f(x_2)$, there is a number $x = c$ between x_1 and x_2 for which $f(c) = y$, implying that the image of $[a, b]$ under f is all real numbers between the minimum and maximum values of $f(x)$, inclusive, Q.E.D.

Problem Set 2-7

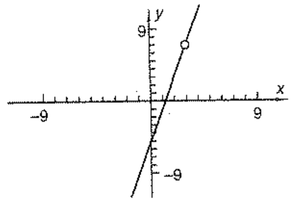
Review Problems

R0. Answers will vary.

R1. a. $f(3) = \dfrac{36 - 51 + 15}{3 - 3} = \dfrac{0}{0}$

Indeterminate form

b. $f(x) = 4x - 5, x \neq 3$



At $x = 3$ there is a removable discontinuity.

c. For 0.01, keep x within 0.0025 unit of 3. For 0.0001, keep x within 0.000025 unit of 3. To keep $f(x)$ within ε unit of 7, keep x within $\frac{1}{4}\varepsilon$ unit of 3.

R2. a. $L = \lim_{x \to c} f(x)$ if and only if for any number $\varepsilon > 0$, no matter how small, there is a number $\delta > 0$ such that if x is within δ units of c, but $x \neq c$, then $f(x)$ is within ε units of L.

b. $\lim_{x \to 1} f(x) = 2$
$\lim_{x \to 2} f(x)$ does not exist.
$\lim_{x \to 3} f(x) = 4$
$\lim_{x \to 4} f(x)$ does not exist.
$\lim_{x \to 5} f(x) = 3$

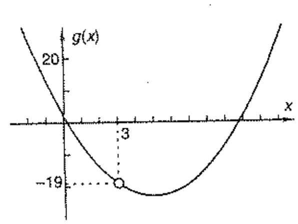
c. $\lim_{x \to 2} f(x) = 3$
Maximum δ: 0.6 or 0.7

d. The left side of $x = 2$ is the more restrictive.
Let $2 + \sqrt{x-1} = 3 - 0.4 = 2.6$.
$\therefore x = 1 + 0.6^2 = 1.36$
$\therefore$ maximum value of δ is $2 - 1.36 = 0.64$.

e. Let $f(x) = 3 - \varepsilon$.
$2 + \sqrt{x-1} = 3 - \varepsilon$
$x = (1 - \varepsilon)^2 + 1$
Let $\delta = 2 - [(1 - \varepsilon)^2 + 1] = 1 - (1 - \varepsilon)^2$, which is positive for all positive $\varepsilon < 1$. If $\varepsilon \geq 1$, simply take $\delta = 1$. Then δ will be positive for *all* $\varepsilon > 0$.

R3. a. See the limit property statements in the text.

b. •



• The limit of a quotient property does not apply because the limit of the denominator is zero.

• $g(x) = \dfrac{(x-3)(x^2 - 10x + 2)}{x - 3}$
$g(x) = x^2 - 10x + 2, x \neq 3$

You can cancel the $(x - 3)$ because the definition of limit says "but not *equal* to 3."

• $\lim_{x \to 3} g(x)$
$= \lim_{x \to 3} x^2 + \lim_{x \to 3} (-10x) + \lim_{x \to 3} 2$ Limit of a sum
$= \lim_{x \to 3} x \cdot \lim_{x \to 3} x - 10 \lim_{x \to 3} x + 2$ Limit of a product, limit of a constant times a function, limit of a constant
$= 3 \cdot 3 - 10(3) + 2$ Limit of x
$= -19$, which agrees with the graph.

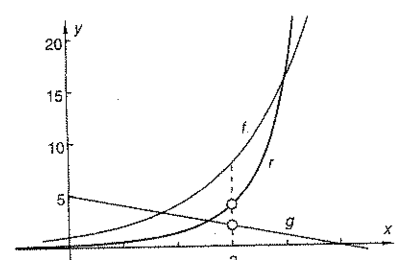
c. • $f(x) = 2^x$,
$g(x) = \dfrac{x^2 - 8x + 15}{3 - x} = \dfrac{(x-3)(x-5)}{3 - x}$
$= -x + 5, x \neq 3$
$\lim_{x \to 3} f(x) = 8, \ \lim_{x \to 3} g(x) = 2$

• $p(x) = f(x) \cdot g(x)$
$\lim_{x \to 3} p(x) = 8 \cdot 2 = 16$

x	$p(x)$
2.997	15.9907...
2.998	15.9938...
2.999	15.9969...
3	undefined
3.001	16.0030...
3.002	16.0061...
3.003	16.0092...

All these $p(x)$ values are close to 16.

• $r(x) = \dfrac{f(x)}{g(x)}$
$\lim_{x \to 3} r(x) = \dfrac{8}{2} = 4$



d. For 5 to 5.1 s: average velocity $= -15.5$ m/s.
Average velocity $= \dfrac{f(t) - f(5)}{t - 5} =$
$\dfrac{35t - 5t^2 - 50}{t - 5} = \dfrac{-5(t-2)(t-5)}{t - 5} =$

$-5(t-2)$, for $t \neq 5$. Instantaneous velocity = limit $= -5(5-2) = -15$ m/s.
The rate is negative, so the distance above the starting point is getting smaller, which means the rock is going down.
Instantaneous velocity is a derivative.

R4. a. • f is continuous at $x = c$ if and only if

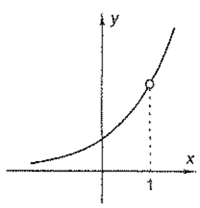
1. $f(c)$ exists
2. $\lim_{x\to c} f(x)$ exists
3. $\lim_{x\to c} f(x) = f(c)$

• f is continuous on $[a, b]$ if and only if f is continuous at every point in (a, b), and $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$.

b.

c	$f(c)$	$\lim_{x\to c^-} f(x)$	$\lim_{x\to c^+} f(x)$	$\lim_{x\to c} f(x)$	Continuous?
1	none	none	none	none	infinite
2	1	3	3	3	removable
3	5	2	5	none	step
4	3	3	3	3	continuous
5	1	1	1	1	continuous

c. i.



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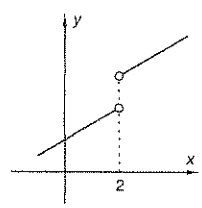

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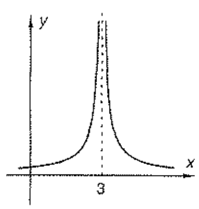

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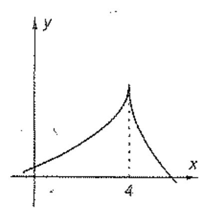

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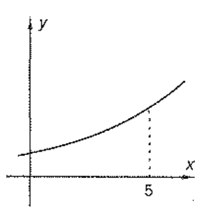

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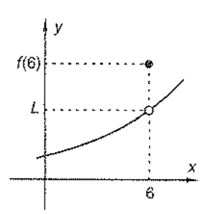
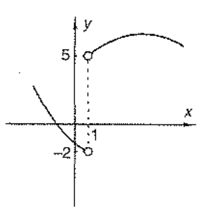
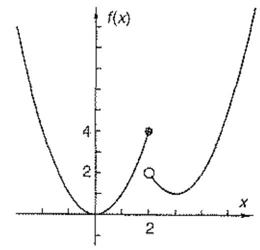


vii.



d.



The left limit is 4 and the right limit is 2, so f is discontinuous at $x = 2$, Q.E.D.
Let $2^2 = 2^2 - 6(2) + k$.
$\therefore k = 12$

R5. a. $\lim_{x\to 4} f(x) = \infty$ means that $f(x)$ can be kept arbitrarily far from 0 on the positive side just by keeping x close enough to 4, but not equal to 4.
$\lim_{x\to\infty} f(x) = 5$ means that $f(x)$ can be made to stay arbitrarily close to 5 just by keeping x large enough in the positive direction.

b. • $\lim_{x\to-\infty} f(x)$ does not exist.

• $\lim_{x\to-2} f(x) = 1$

• $\lim_{x\to 2^-} f(x) = \infty$

• $\lim_{x\to 2^+} f(x) = -\infty$

• $\lim_{x\to\infty} f(x) = 2$

c. $f(x) = 6 - 2^{-x}$
$\lim_{x\to\infty} f(x) = 6$
$f(x) = 5.999 = 6 - 2^{-x}$
$2^{-x} = 0.001$
$$x = -\frac{\log 0.001}{\log 2}$$
$x = 9.965...$

x	$f(x)$
10	5.999023...
20	5.999999046...
30	5.99999999907...

All of these $f(x)$ values are within 0.001 of 6.

d. $g(x) = x^{-2}$
$\lim_{x\to 0} g(x) = \infty$
$g(x) = 10^6 = x^{-2}$
$x^2 = 10^{-6}$
$x = 10^{-3}$

x	$g(x)$
0.0009	$1.2345... \cdot 10^6$
0.0005	4,000,000
0.0001	$1 \cdot 10^8$

All of these $g(x)$ values are larger than 1,000,000.

e. $v(t) = 40 + 6\sqrt{t}$

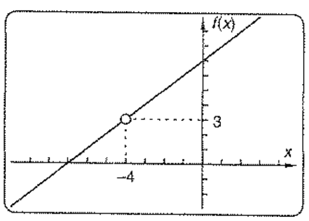
n	Trapezoidal Rule
50	467.9074...
100	467.9669...
200	467.9882...
400	467.9958...

The limit of these sums seems to be 468.

By exploration,
$T_{222} = 467.98995\ldots$
$T_{223} = 467.99002\ldots$
$\therefore D = 223$

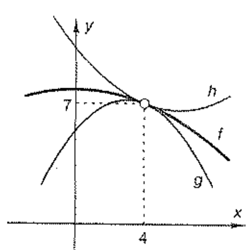
R6. a. See the text statement of the intermediate value theorem.
The basis is the completeness axiom.
See the text statement of the extreme value theorem.
The word is *corollary*.

b. $f(x) = -x^3 + 5x^2 - 10x + 20$
$f(3) = 8, f(4) = -4$
So $f(x) = 0$ for some x between 3 and 4 by the intermediate value theorem.
The property is continuity.
The value of x is approximately 3.7553.

c. 

$f(-6) = 1$ and $f(-2) = 5$ by tracing on the graph or by simplifying the fraction to get $f(x) = x + 7$, then substituting. You will *not* always get a value of x if y is between 1 and 5. If you pick $y = 3$, there is no value of x. This fact does not contradict the intermediate value theorem. Function f does not meet the continuity hypothesis of the theorem.

Concept Problems

C1. 

Conjecture: $\lim_{x\to 4} f(x) = 7$

C2. $f(1) = 1^2 - 6 \cdot 1 + 9 = 4$
As $x \to 1$ from the left, $f(x) \to 1^2 + 3 = 4$.
As $x \to 1$ from the right, $f(x) \to 1^2 - 6 + 9 = 4$.
$\therefore \lim_{x\to 1} f(x) = 4 = f(1)$
$\therefore f$ is continuous at $x = 4$, Q.E.D.
For the derivative, from the left side,
$\dfrac{f(x)-f(1)}{x-1} = \dfrac{x^2+3-4}{x-1} = \dfrac{(x-1)(x+1)}{x-1} =$
$x + 1, x \neq 1$
$\therefore \lim_{x\to 1^-} f'(x) = 1 + 1 = 2$
For the derivative, from the right side,
$\dfrac{f(x)-f(1)}{x-1} = \dfrac{x^2-6x+9-4}{x-1} = \dfrac{(x-1)(x-5)}{x-1} =$
$x - 5, x \neq 1$
$\therefore \lim_{x\to 1^+} f'(x) = 1 - 5 = -4$
So f is continuous at $x = 1$, but does not have a value for the derivative there because the rate of change jumps abruptly from 2 to -4 at $x = 1$. In general, if a function has a cusp at a point, then the derivative does not exist, but the function is still continuous.

C3. The graph is a $y = x^2$ parabola with a step discontinuity at $x = 1$. (Use the "rise-run" property. Start at the vertex. Then run 1, rise 1; run 1, rise 3; run 1, rise 5; . . . . Ignore the discontinuity at first.) To create the discontinuity, use the signum function with argument $(x - 1)$. Because there is no value for $f(1)$, the absolute value form of the signum function can be used.

$$y = x^2 + 2 - \frac{|x-1|}{x-1}$$

C4. The quantity $|f(x) - L|$ is the distance between $f(x)$ and L. If this distance is less than ε, then $f(x)$ is within ε units of L. The quantity $|x - c|$ is the distance between x and c. The right part of the inequality, $|x - c| < \delta$, says that x is within δ units of c. The left part, $0 < |x - c|$, says that x does not equal c. Thus, this definition of limit is equivalent to the other definition.

Chapter Test

T1. f is continuous at $x = c$ if and only if
1. $f(c)$ exists
2. $\lim_{x\to c} f(x)$ exists
3. $\lim_{x\to c} f(x) = f(c)$

f is continuous on $[a, b]$ if and only if f is continuous at all points in (a, b), and $\lim_{x\to a^+} f(x) = f(a)$ and $\lim_{x\to b^-} f(x) = f(b)$.