Problem Set 6-7

Review Problems

R0. Answers will vary.

R1. a. $dM/dt = 0.06M \Rightarrow M^{-1}\,dM = 0.06\,dt$

$$\therefore \int_{100}^{x} M^{-1}\,dM = \int_0^5 0.06\,dt, \text{ Q.E.D.}$$

$$\int_0^5 0.06\,dt = 0.06t\Big|_0^5 = 0.3$$

b. Solving numerically for x in

$$\int_{100}^{x} M^{-1}\,dM = 0.3$$

gives $x \approx 134.9858\ldots$.

c. There will be \$134.99 in the account, so the interest will be \$34.99.

R2. a. Integrating x^{-1} by the power rule results in division by zero: $\dfrac{x^{-1+1}}{-1+1} + C$.

b. If $g(x) = \int_a^x f(t)\,dt$ and $f(x)$ is continuous in a neighborhood of a, then $g'(x) = f(x)$.

$$\ln x = \int_1^x \frac{1}{t}\,dt$$

$$\frac{d}{dx}(\ln x) = \frac{d}{dx}\left(\int_1^x \frac{1}{t}\,dt\right) = \frac{1}{x}$$

c. i. $y = (\ln 5x)^3 \Rightarrow y' = (3/x)(\ln 5x)^2$

ii. $f(x) = \ln x^9 = 9 \ln x \Rightarrow f'(x) = 9/x$

iii. $y = \csc (\ln x) \Rightarrow$
$y' = -\csc (\ln x) \cot (\ln x) \cdot (1/x)$

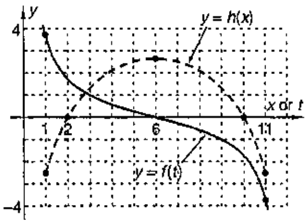
iv. $g(x) = \int_1^{x^2} \csc t\,dt \Rightarrow g'(x) = 2x \csc x^2$

d. i. $\int \dfrac{\sec x \tan x}{\sec x}\,dx = \int \dfrac{1}{\sec x} \sec x \tan x\,dx$
$= \ln|\sec x| + C$

ii. $\int_{-2}^{-3} \dfrac{10}{x}\,dx = 10 \ln|x|\Big|_{-2}^{-3}$
$= 10 \ln|-3| - 10 \ln|-2|$
$= 10(\ln 3 - \ln 2) = 4.054651\ldots$

iii. $\int x^2(x^3-4)^{-1}\,dx = \dfrac{1}{3}\int (x^3-4)^{-1}(3x^2\,dx)$
$= \dfrac{1}{3}\ln|x^3-4| + C$

e. By finding areas, $h(1) = -2.5$, $h(2) = 0$, $h(6) = 2.7$, $h(10) = 0$, and $h(11) = -2.5$.



f. i. $y(100) \approx 70$ names; 70% remembered
$y(1) = 1$ name; 100% remembered

ii. $y' = \dfrac{101}{100 + x}$

$y'(100) = 101/(200) = 0.505$ names/person
$y'(1) = 101/101 = 1$ name/person

iii. Paula has probably not forgotten any names as long as $x - y < 0.5$. After meeting 11 people, she remembers about $10.53... \approx 11$ names, but after meeting 12 people, she remembers about $11.44... \approx 11$ names.

R3. a. i. See the text for the definition of *logarithm.*

ii. See the text for the definition of $\ln x$.

iii. See the text for the statement of the uniqueness theorem.

iv. See the text for the proof.

v. See the solution to Problem 10 in Lesson 6-3.

b. i. $e = \lim_{n \to 0} (1 + n)^{1/n}$ or $e = \lim_{n \to \infty} (1 + 1/n)^n$

ii. $\log_b x = \dfrac{\ln x}{\ln b}$

c. i. $y = \log_4 x = \dfrac{\ln x}{\ln 4} \Rightarrow y' = \dfrac{1}{x \ln 4}$

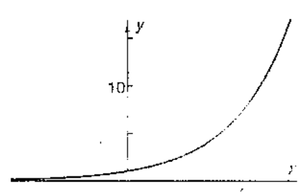
ii. $f(x) = \log_2 (\cos x) = \dfrac{\ln(\cos x)}{\ln 2} \Rightarrow$

$$f'(x) = \frac{1}{(\cos x)(\ln 2)} \cdot (-\sin x)$$

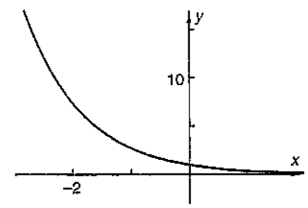
$$= -\frac{\tan x}{\ln 2}$$

iii. $y = \log_5 9^x = x \log_5 9 \Rightarrow y' = \log_5 9$

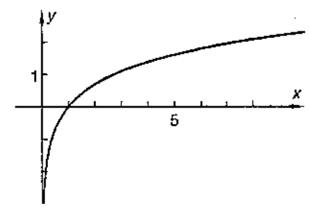
R4. a. i.



ii.



iii.



b. i. $f(x) = x^{1.4} e^{5x} \Rightarrow$

$f'(x) = 1.4x^{0.4} e^{5x} + 5x^{1.4} e^{5x}$

ii. $g(x) = \sin(e^{-2x}) \Rightarrow g'(x) = -2e^{-2x} \cos(e^{-2x})$

iii. $\dfrac{d}{dx}(e^{\ln x}) = \dfrac{d}{dx}(x) = 1$

iv. $y = 100^x \Rightarrow y' = (\ln 100)100^x$

v. $f(x) = 3.7 \cdot 10^{0.2x} \Rightarrow$

$f'(x) = 0.74 \ln 10 \cdot 10^{0.2x}$

vi. $r(t) = t^{\tan t} \Rightarrow \ln r = \tan t \ln t \Rightarrow$

$$\frac{1}{r} r' = \sec^2 t \ln t + \frac{\tan t}{t} \Rightarrow$$

$$r' = t^{\tan t} \left(\sec^2 t \ln t + \frac{\tan t}{t} \right)$$

c. $y = (5x - 7)^3 (3x + 1)^5 \Rightarrow$

$\ln y = 3 \ln (5x - 7) + 5 \ln (3x + 1) \Rightarrow$

$$\frac{1}{y} y' = \frac{15}{5x - 7} + \frac{15}{3x + 1} \Rightarrow$$

$$y' = \left(\frac{15}{5x - 7} + \frac{15}{3x + 1} \right)(5x - 7)^3 (3x + 1)^5$$

$$= (120x - 90)(5x - 7)^2 (3x + 1)^4$$

d. i. $\int 10e^{-2x}\, dx = -5e^{-2x} + C$

ii. $\int e^{\cos x} \sin x\, dx = -e^{\cos x} + C$

iii. $\int_{-2}^{2} e^{-0.1x}\, dx = -10e^{-0.1x} \Big|_{-2}^{2}$

$$= -10e^{-0.2} + 10e^{0.2} = 4.02672...$$

iv. $\int 10^{0.2x}\, dx = \dfrac{10^{0.2x}}{0.2 \ln 10} + C$

e. i. The exposure is the product of $C(t)$ and t, where $C(t)$ varies. Thus, a definite integral must be used.

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ii. $E(x) = \int_0^x 150e^{-0.16t}\,dt = 937.5(-e^{-0.16x} + 1)$

$E(5) = 937.5(-e^{-0.8} + 1) =$ 516.25... ppm · days $E(10) =$ $937.5(-e^{-1.6} + 1) = 748.22...$ ppm · days

As x grows very large, $E(x)$ seems to approach 937.5.

iii. $E'(x) = 150e^{-0.16x} = C(x)$

$E'(5) = 67.39...$ ppm (or ppm · days per day)

$E'(10) = 30.28...$ ppm

f. i. From Figure 6-7d, the maximum concentration is about 150 ppm at about 2 hours. (These values can be found more precisely by setting the numerical or algebraic derivative equal to zero, solving to get $t = -1/\ln 0.6 = 1.9576...$. Then $C(1.9576...) = -200/(e \ln 0.6) =$ 144.0332... .)

ii. $C(t) = 200t \cdot 0.6^t$

$C'(t) = 200t \cdot 0.6^t \ln 0.6 + 200 \cdot 0.6^t$

$C'(1) = 200 \cdot 0.6^1(\ln 0.6 + 1) = 58.70...$

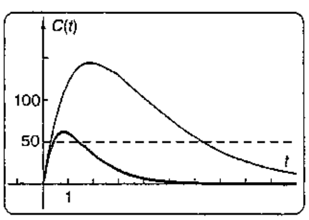
$C'(5) = 200 \cdot 0.6^5(5 \ln 0.6 + 1)$

$= -24.16... < 0$

$C(t)$ is increasing at about 58.7 ppm/h when $t = 1$ and decreasing at about 24.2 ppm/h when $t = 5$. The concentration is increasing if $C'(t)$ is positive and decreasing if it is negative.

iii. Solving $50 = 200t \cdot 0.6t$ numerically for t gives $t \approx 0.2899...$ and $t \approx 6.3245...$. So $C(t) > 50$ for $6.3245... - 0.2899... =$ 6.03... , or about 6 hours.

iv. $C_1(t) = 200t \cdot 0.3^t$



From the graph, the maximum is about 60 ppm around $t = 1$. (Exactly, $t = -1/\ln 0.3 =$ 0.8305... , for which $C(0.8305...) =$ $-200/(e \ln 0.3) = 61.11092... \approx 61.1$ ppm.) Repeating the computations of part iii gives $C(t) > 50$ for $0.409... < t < 1.473...$, or for about 1.06 hours.

In conclusion, the concentration peaks sooner at a lower concentration and stays above 50 ppm for a much shorter time.

$\ln 0.5 = -0.025t$

R5. a. $\lim_{x\to\infty} \frac{2x^2-3}{7-5x^2} \to \frac{\infty}{-\infty}$

$= \lim_{x\to\infty} \frac{4x}{-10x} = -\frac{2}{5}$

b. $\lim_{x\to 0} \frac{x^2 - \cos x + 1}{e^x - x - 1} \to \frac{0}{0}$

$= \lim_{x\to 0} \frac{2x + \sin x}{e^x - 1} \to \frac{0}{0}$

$= \lim_{x\to 0} \frac{2 + \cos x}{e^x} = \frac{2+1}{1} = 3$

c. $\lim_{x\to\infty} x^3 e^{-x} \to \infty \cdot 0$

$= \lim_{x\to\infty} \frac{x^3}{e^x} \to \frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{3x^2}{e^x} \to \frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{6x}{e^x} \to \frac{\infty}{\infty}$

$= \lim_{x\to\infty} \frac{6}{e^x} = 0$ (Form: $6/\infty$)

d. $L = \lim_{x\to 1} x^{\tan(\pi x/2)} \to 1^\infty$

$\ln L = \lim_{x\to 1} [\tan(\pi x/2) \cdot \ln x]$

$= \lim_{x\to 1} \frac{\ln x}{\cot(\pi x/2)} \to \frac{0}{0}$

$= \lim_{x\to 1} \frac{1/x}{-(\pi/2)\csc^2 \pi x/2} = \frac{1}{-\pi/2} = \frac{-2}{\pi}$

$\therefore L = e^{-2/\pi} = 0.529077...$

e. $\lim_{x\to 2} 3x^4 = 48$

f. $\lim_{x\to\pi/2} (\tan^2 x - \sec^2 x) = \lim_{x\to\pi/2} (1) = 1$

g. Examples of indeterminate forms:

$0/0$, ∞/∞, $0 \cdot \infty$, 0^0, 1^∞, ∞^0, $\infty - \infty$

R6. a. i. $y = \ln(\sin^4 7x) = 4 \ln \sin 7x \Rightarrow$

$y' = 4(1/\sin 7x) \cdot \cos 7x \cdot 7 = 28 \cot 7x$

ii. $y = x^{-3}e^{2x} \Rightarrow$

$y' = -3x^{-4} \cdot e^{2x} + x^{-3} \cdot 2e^{2x}$

$= x^{-4}e^{2x}(2x - 3)$

iii. $y = \cos(2^x) \Rightarrow y' = -\sin(2^x) \cdot 2^x \ln 2$

iv. $y = \log_3 x^4 = \frac{4 \ln x}{\ln 3} \Rightarrow y' = \frac{4}{x \ln 3}$

b. i. $\int e^{-1.7x}\,dx = (-1/1.7)\,e^{-1.7x} + C$

ii. $\int 2^{\sec x} \sec x \tan x\,dx$

$= \int e^{\ln 2 \sec x} \sec x \tan x\,dx$

R6 ii cont.

$$= (1/\ln 2) \int e^{\ln 2 \sec x} (\ln 2 \cdot \sec x \tan x \, dx)$$

$$= (1/\ln 2)\, e^{\sec x \ln 2} + C = (1/\ln 2)\, 2^{\sec x} + C$$

iii. $\int (5 + \sin x)^{-1} \cos x \, dx = \ln (5 + \sin x) + C$

(No absolute value is needed this time.)

iv. $\int_1^5 \frac{1}{z} dz = \ln 5$ (By definition!)

c. i. $\lim_{x \to 0} \frac{\tan 3x}{x^2} \to \frac{0}{0} = \lim_{x \to 0} \frac{3 \sec^2 3x}{2x} \to \frac{3}{0} = \infty$

ii. $L = \lim_{x \to \infty} (1 - 3/x)^x \to 1^\infty$

$$\ln L = \lim_{x \to \infty} [x \ln (1 - 3/x)] \to \infty \cdot 0$$

$$= \lim_{x \to \infty} \frac{\ln (1 - 3x^{-1})}{x^{-1}} \to \frac{0}{0}$$

$$= \lim_{x \to \infty} \frac{1/(1 - 3x^{-1}) \cdot 3x^{-2}}{-x^{-2}}$$

$$= \lim_{x \to 0} \frac{-3}{1 - 3x^{-1}} = -3$$

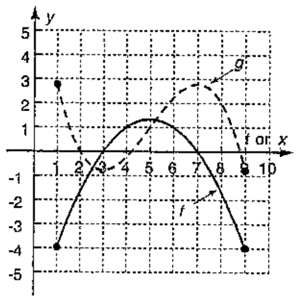
$$\therefore L = e^{-3} = 0.049787\ldots$$

T15. $\int_0^2 5^x\,dx = \frac{1}{\ln 5}5^x\Big|_0^2 = \frac{1}{\ln 5}(25-1)$
$= 14.9120\ldots$

T16. $\lim_{x\to\infty}\frac{5-3x}{\ln 4x} \to \frac{-\infty}{\infty}$
$= \lim_{x\to\infty}\frac{-3}{[1/(4x)]\cdot 4} = \lim_{x\to\infty}(-3x) = -\infty$

T17. $L = \lim_{x\to\pi/2^-}(\tan x)^{\cot x} \to \infty^0$
$\ln L = \lim_{x\to\pi/2^-}[\cot x\cdot\ln(\tan x)] \to 0\cdot\infty$
$= \lim_{x\to\pi/2^-}\frac{\ln(\tan x)}{\tan x} \to \frac{\infty}{\infty}$
$= \lim_{x\to\pi/2^-}\frac{(1/\tan x)\cdot\sec^2 x}{\sec^2 x} = \lim_{x\to\pi/2^-}\cot x = 0$
$\therefore L = e^0 = 1$

T18. a.



b. $h(x) = \int_2^{3x-5} f(t)\,dt \Rightarrow h'(x) = f(3x-5)\cdot 3,$
$h'(3) = f(9-5)\cdot 3 = f(4)\cdot 3 = 1\cdot 3 = 3$

T19. $\ln x = \int_1^x \frac{1}{t}\,dt$, so $\ln 1.8 = \int_1^{1.8}\frac{1}{t}\,dt$

$M_4 = 0.2\left(\frac{1}{1.1}+\frac{1}{1.3}+\frac{1}{1.5}+\frac{1}{1.7}\right) = 0.58664\ldots$

From calculator, $\ln 1.8 = 0.58778\ldots$.

T20. $g(x) = \int_2^{x^2}\sin t\,dt = -\cos t\Big|_2^{x^2}$

$= -\cos x^2 + \cos 2 \Rightarrow g'(x) = 2x\sin x^2$

$g'(x) = \frac{d}{dx}\int_2^{x^2}\sin t\,dt = 2x\sin x^2$

T21. Let $h(x) = f(x) - g(x)$.
Then $h(a) = f(a) - g(a) = 0$ and
$h(b) = f(b) - g(b) \neq 0$.
$\therefore \frac{h(b)-h(a)}{b-a} \neq 0$
By the mean value theorem, there is a number c between a and b such that

$h'(c) = \frac{h(b)-h(a)}{b-a}$.

$\therefore h'(c) \neq$

But $h'(x) = f'(x) - g'(x)$, which equals 0 for all values of x.

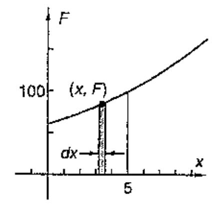
$\therefore h'(c) = 0$

This result thus contradicts the mean value theorem, Q.E.D.

T22. a. $F(x) = 60e^{0.1x} \Rightarrow F'(x) = 6e^{0.1x}$ so
$F'(5) = 6e^{0.5} = 9.8923\ldots$ lb/ft
$F'(10) = 6e = 16.3096\ldots$ lb/ft

b. Work equals force times displacement. But the force varies at different displacements. Thus, a definite integral has to be used.

c.



$dW = F\,dx = 60\,e^{0.1x}\,dx$
$W = \int_0^5 60e^{0.1x}\,dx$
$= 600e^{0.1x}\Big|_0^5 = 600(e^{0.5}-1)$
$W \approx 389.23$ ft-lb

T23. Answers will vary.

Problem Set 6-8

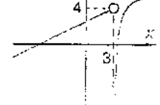
Cumulative Review, Chapters 1–6

1. $f(x) = 2^x$
$f'(3) \approx \frac{2^{3.1}-2^{2.9}}{0.2} = 5.549618\ldots$

2. There are about 10.0 squares, each 20 units.
$\therefore \int_{10}^{50} g(x)\,dx \approx 200$
(Function is $g(x) = 2 + 0.1x + \sin\frac{2\pi}{15}x$, so exact answer is 200.)

3. $L = \lim_{x\to c} f(x)$ if and only if for any $\varepsilon > 0$, there is a $\delta > 0$ such that if x is within δ units of c but not equal to c, $f(x)$ is within ε units of L.

4. Answers may vary.



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Chapter Test

T1. $\ln x = \int_1^x \frac{1}{t}\,dt$

T2. $e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n$ or $e = \lim_{n\to 0}(1+n)^{1/n}$

T3. If $g(x) = \int_a^x f(t)\,dt$ and $f(t)$ is continuous in a neighborhood of a, then $g'(x) = f(x)$.

T4. If (1) $f'(x) = g'(x)$ for all x in the domain and (2) $f(a) = g(a)$ for some a in the domain, then $f(x) = g(x)$ for all x in the domain.

T5. Prove that $\ln x = \log_e x$ for all $x > 0$.

Proof:

Let $f(x) = \ln x$, and $g(x) = \log_e x$.
$f'(x) = 1/x$ and
$g'(x) = (1/x)\cdot\log_e e = (1/x)\cdot 1 = 1/x$
$\therefore f'(x) = g'(x)$ for all $x > 0$
$f(1) = \ln 1 = 0$ and $g(1) = \log_e 1 = 0$
$\therefore f(1) = g(1)$
$\therefore$ by the uniqueness theorem,
$f(x) = g(x)$ for all $x > 0$.
$\therefore \ln x = \log_e x$ for all $x > 0$, Q.E.D.

T6. $f(x) = \ln(x^3e^x)$

a. $f'(x) = \frac{1}{x^3e^x}\cdot(3x^2e^x + x^3e^x) = 3/x + 1$

b. $f(x) = 3\ln x + x\ln e = 3\ln x + x \Rightarrow$
$f'(x) = 3/x + 1$ (Checks.)

T7. $y = e^{2x}\ln x^3 = 3e^{2x}\ln x \Rightarrow$
$y' = 6e^{2x}\cdot\ln x + 3e^{2x}\cdot(1/x)$
$= 3e^{2x}(2\ln x + 1/x)$

T8. $v = \ln(\cos 10x) \Rightarrow$
$v' = 1/(\cos 10x)\cdot(-10\sin 10x) = -10\tan 10x$

T9. $f(x) = (\log_2 4x)^7 = [(\ln 4x)/(\ln 2)]^7 \Rightarrow$
$f'(x) = 7[(\ln 4x)/(\ln 2)]^6\cdot[(1/4x)\cdot 4\cdot(1/\ln 2)]$
$= \frac{7(\log_2 4x)^6}{x\ln 2}$

T10. $t(x) = \ln(\cos^2 x + \sin^2 x) = \ln 1 = 0 \Rightarrow$
$t'(x) = 0$

T11. $p(x) = \int_1^{\ln x} e^t \sin t\,dt \Rightarrow$
$p'(x) = e^{\ln x}\sin\ln x\cdot 1/x = \sin\ln x$

T12. $\int e^{5x}\,dx = \frac{1}{5}e^{5x} + C$

T13. $\int(\ln x)^6(dx/x) = \frac{1}{7}(\ln x)^7 + C$

T14. $\int \sec 5x\,dx = \frac{1}{5}\ln|\sec 5x + \tan 5x| + C$