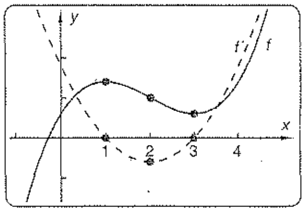
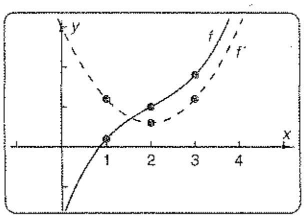
Chapter 8—The Calculus of Plane and Solid Figures

Problem Set 8-1

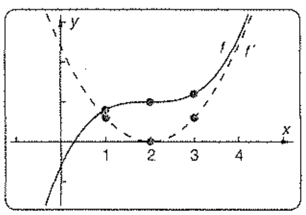
1. $f(x) = x^3 - 6x^2 + 9x + 3$
   $f'(x) = 3x^2 - 12x + 9$



   $g(x) = x^3 - 6x^2 + 15x - 9$
   $g'(x) = 3x^2 - 12x + 15$



   $h(x) = x^3 - 6x^2 + 12x - 3$
   $h'(x) = 3x^2 - 12x + 12$



   Positive derivative ⇒ increasing function
   Negative derivative ⇒ decreasing function
   Zero derivative ⇒ function could be at a high point or a low point, but not always.

2. The functions have vertex points at values of x where their derivatives change sign. If the derivative is never zero, as for function g, the function graph has no vertex points. If the derivative is zero but does not change sign, as for function h, the function graph just levels off, then continues in the same direction, with no vertex.

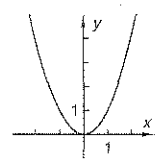
3. $g''(x) = (d/dx)(3x^2 - 12x + 15) = 6x - 12$
   $h''(x) = (d/dx)(3x^2 - 12x + 12) = 6x - 12$
   All the second derivatives are the same!

4. The curves are concave up where the second derivative is positive and concave down where the second derivative is negative.

5. Points of inflection occur where the first derivative graph reaches a minimum.
   Points of inflection occur where the second derivative graph crosses the x-axis.

Problem Set 8-2

Q1.



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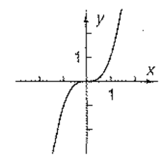

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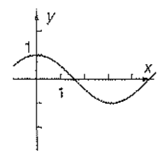

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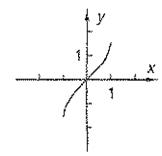

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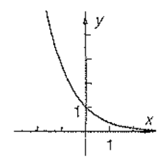

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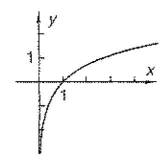

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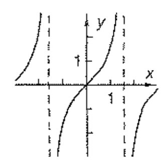

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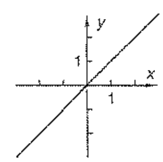

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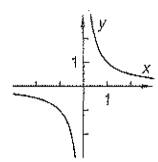

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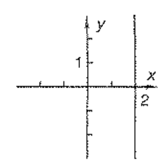

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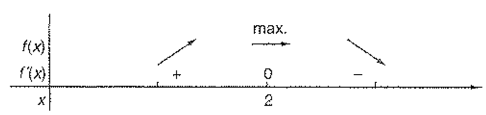

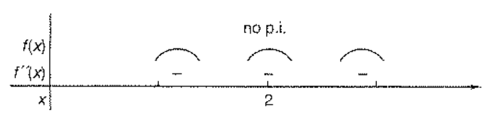

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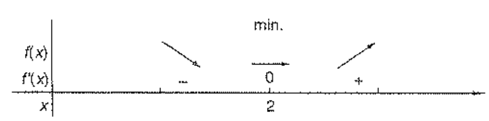

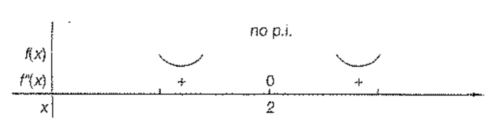

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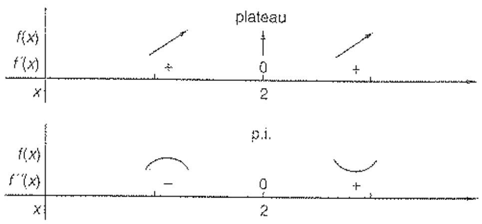

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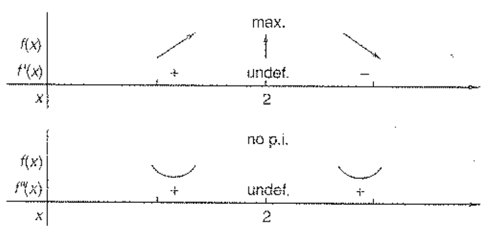

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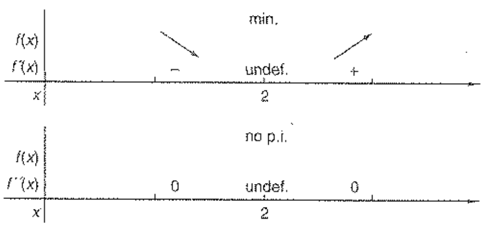

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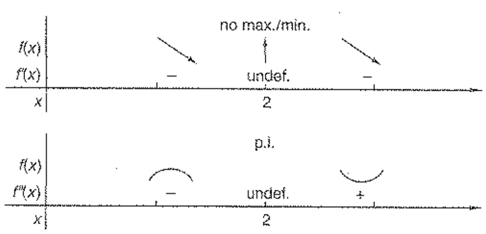

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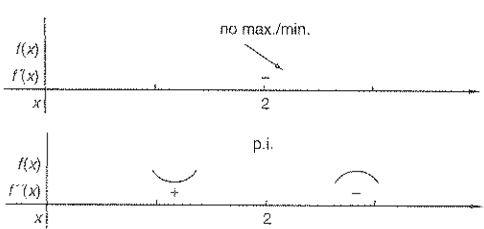

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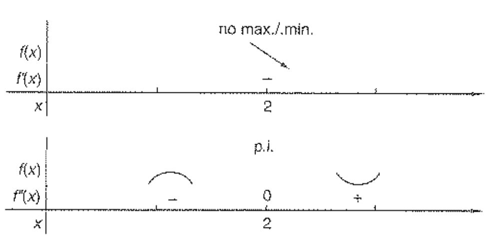

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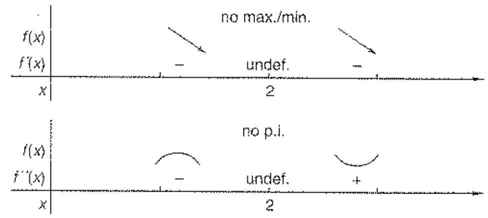

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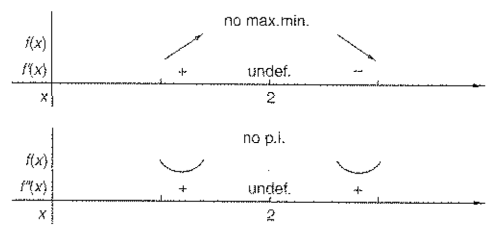

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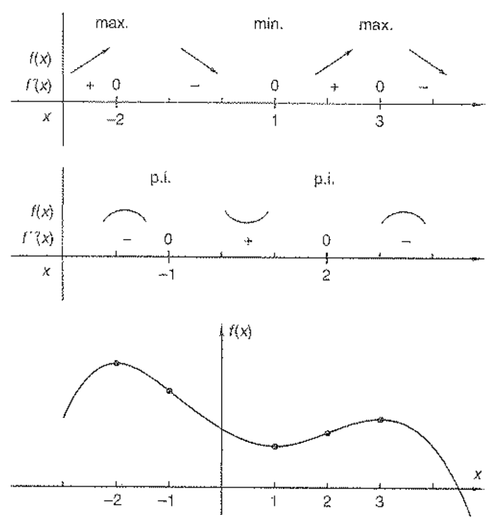

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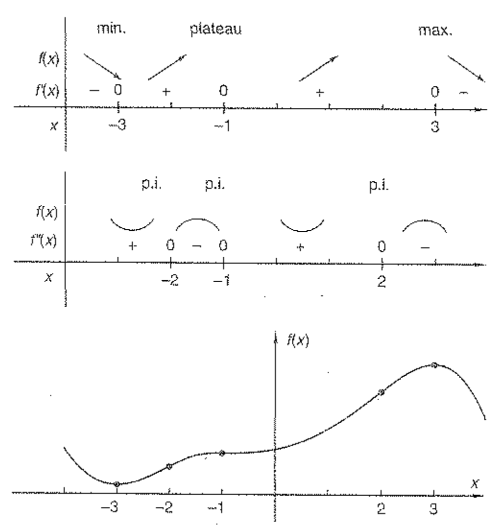

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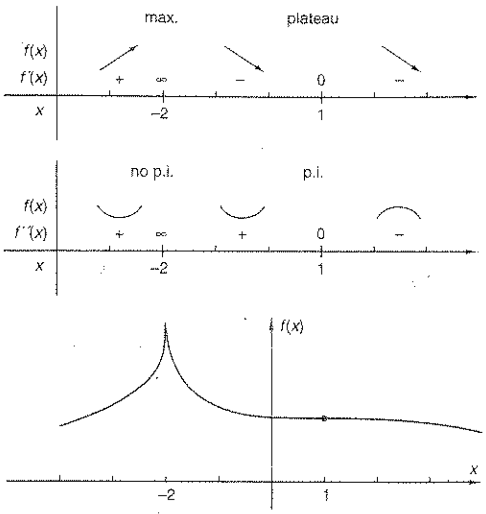

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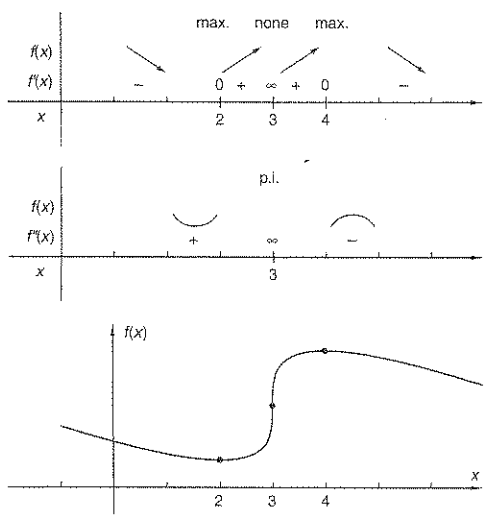

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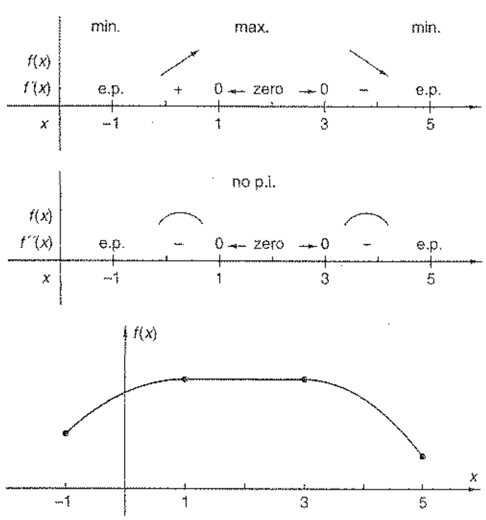

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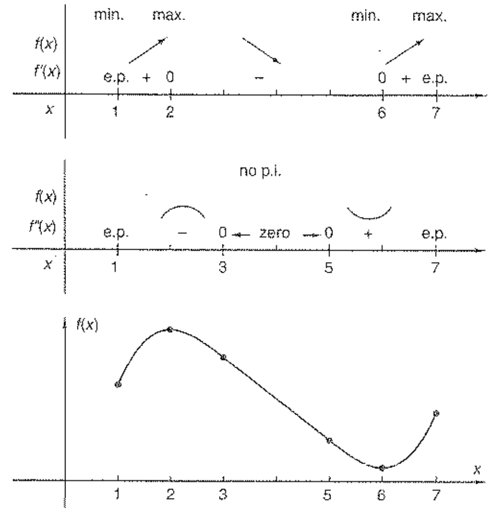

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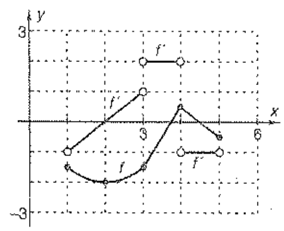

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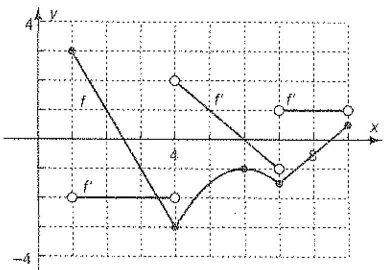

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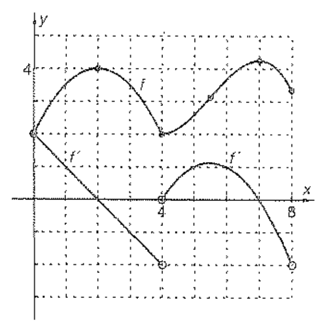

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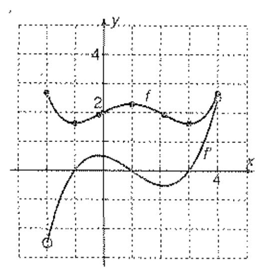


21. $f(x) = 3e^x - xe^x$

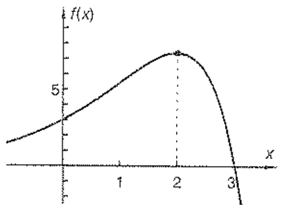
$f'(x) = 3e^x - e^x - xe^x = e^x(2 - x)$

$f'(2) = e^2(2 - 2) = 0 \Rightarrow$ critical point at $x = 2$

$f''(x) = 2e^x - e^x - xe^x = e^x(1 - x)$

$f''(2) = e^2(1 - 2) = -7.3890... < 0$

$\therefore$ local maximum at $x = 2$



The graph confirms a maximum at $x = 2$.

22. $f(x) = -\sin \dfrac{\pi}{4} x$

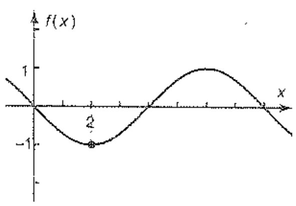
$f'(x) = -\dfrac{\pi}{4} \cos \dfrac{\pi}{4} x$

$f'(2) = -\dfrac{\pi}{4} \cos \dfrac{\pi}{4}(2) = 0 \Rightarrow$ critical point at $x = 2$

$f''(x) = \dfrac{\pi^2}{16} \sin \dfrac{\pi}{4} x$

$f''(2) = \dfrac{\pi^2}{16} \sin \dfrac{\pi}{4}(2) = 0.6168\ldots > 0$

$\therefore$ local minimum at $x = 2$



The graph confirms a minimum at $x = 2$.

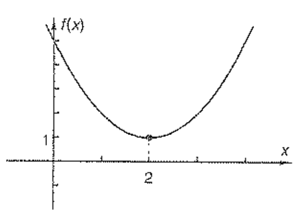
23. $f(x) = (2 - x)^2 + 1$

$f'(x) = -2(2 - x)$

$f'(2) = -2(2 - 2) = 0 \Rightarrow$ critical point at $x = 2$

$f''(x) = 2 \Rightarrow f''(2) = 2 > 0$

$\therefore$ local minimum at $x = 2$



The graph confirms a minimum at $x = 2$.

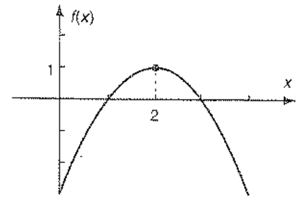
24. $f(x) = -(x - 2)^2 + 1$

$f'(x) = -2(x - 2)$

$f'(2) = -2(2 - 2) = 0 \Rightarrow$ critical point at $x = 2$

$f''(x) = -2 \Rightarrow f''(2) = -2 < 0$

$\therefore$ local maximum at $x = 2$



The graph confirms a maximum at $x = 2$.

25. $f(x) = (x - 2)^3 + 1$

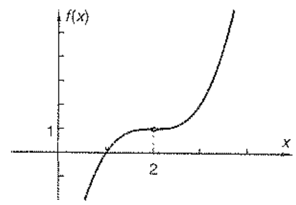
$f'(x) = 3(x - 2)^2$

$f'(2) = 3(2 - 2)^2 = 0 \Rightarrow$ critical point at $x = 2$

$f''(x) = 6(x - 2)$

$f''(2) = 6(2 - 2) = 0$, so the test fails.

$f'(x)$ goes from positive to positive as x increases through 2, so there is a plateau at $x = 2$.



The graph confirms a plateau at $x = 2$.

26. $f(x) = (2 - x)^4 + 1$

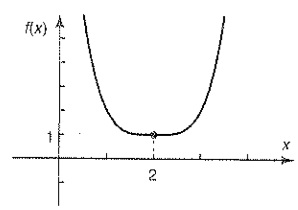
$f'(x) = -4(2 - x)^3$

$f'(2) = -4(2 - 2)^3 = 0 \Rightarrow$ critical point at $x = 2$

$f''(x) = 12(2 - x)^2$

$f''(2) = 12(2 - 2)^2 = 0$, so the test fails.

$f'(x)$ changes from negative to positive as x increases through 2, so there is a local minimum at $x = 2$.



The graph confirms a minimum at $x = 2$.

27. a. $f(x) = 6x^5 - 10x^3$

$f'(x) = 30x^4 - 30x^2 = 30x^2(x + 1)(x - 1)$

$f'(x) = 0 \Leftrightarrow x = -1, 0,$ or 1 (critical points for $f(x)$)

$f''(x) = 120x^3 - 60x = 60x(\sqrt{2}x + 1)(\sqrt{2}x - 1)$

$f''(x) = 0 \Leftrightarrow x = 0, \pm\sqrt{1/2}$ (critical points for $f'(x)$)

b. The graph begins after the f-critical point at $x = -1$; the f'-critical point at $x = -\sqrt{1/2}$ is shown, but is hard to see.

c. $f'(x)$ is negative for both $x < 0$ and $x > 0$.

28. a. $f(x) = 0.1x^4 - 3.2x + 7$
$f'(x) = 0.4x^3 - 3.2 = 0.4(x-2)(x^2 + 2x + 4)$
$x^2 + 2x + 4$ has discriminant $= 2^2 - 4 \cdot 4 < 0$, so $f'(x) = 0 \Leftrightarrow x = 2$ (critical point for $f(x)$).
$f''(x) = 1.2x^2$
$f''(x) = 0 \Leftrightarrow x = 0$ (critical point for $f'(x)$)

b. $f''(x)$ does not change sign at $x = 0$. ($f''(x) \ge 0$ for all x)

c. $f''(c) = 0$, but $f'(c) \ne 0$

29. a. $f(x) = xe^{-x}$
$f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x)$
$f'(x) = 0 \Leftrightarrow x = 1$ (critical point for $f(x)$)
$f''(x) = xe^{-x} - 2e^{-x} = e^{-x}(x-2)$
$f''(x) = 0 \Leftrightarrow x = 2$ (critical point for $f'(x)$)

b. Because $f(x)$ approaches its horizontal asymptote ($y = 0$) from above, the graph must be concave up for large x; but the graph is concave down near $x = 1$, and the graph is smooth; somewhere the concavity must change from down to up.

c. No. $e^{-x} \ne 0$ for all x, so $xe^{-x} = 0 \Leftrightarrow x = 0$.

30. a. $f(x) = x^2 \ln x$
$f'(x) = x + 2x \ln x = x(1 + 2 \ln x)$
$f(x)$ and $f'(x)$ are undefined at $x = 0$, so $f'(x) = 0 \Leftrightarrow \ln x = -0.5 \Leftrightarrow x = e^{-0.5} =$ 0.6065... (critical point for $f(x)$).
$f''(x) = 3 + 2 \ln x$
$f''(x) = 0 \Leftrightarrow \ln x = -1.5 \Leftrightarrow x = e^{-1.5} =$ 0.2231... (critical point for $f'(x)$).

b. $\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{x^{-2}} = \lim_{x\to 0^+} \frac{1/x}{-2x^{-3}}$
$= \lim_{x\to 0^+} -0.5x^2 = 0$ by L'Hospital's rule.
$\lim_{x\to 0^-} x^2 \ln x$ does not exist because $x^2 \ln x$ is undefined for $x < 0$.

c. All critical points from part a appear, although the inflection point at $x = e^{-1.5}$ is hard to see on the graph.

31. a. $f(x) = x^{5/3} + 5x^{2/3}$
$$f'(x) = \frac{5}{3}x^{2/3} + \frac{10}{3}x^{-1/3} = \frac{5}{3}x^{-1/3}(x+2)$$
$f'(x) = 0 \Leftrightarrow x = -2$, and $f'(x)$ is undefined at $x = 0$ (critical points for $f(x)$).
$$f''(x) = \frac{10}{9}x^{-1/3} - \frac{10}{9}x^{-4/3} = \frac{10}{9}x^{-4/3}(x-1)$$
$f''(x) = 0 \Leftrightarrow x = 1$ (critical point for $f'(x)$; $f'(0)$ is undefined, so f' has no critical point at $x = 0$).

b. The y-axis ($x = 0$) is a tangent line because the slope approaches $-\infty$ from both sides.

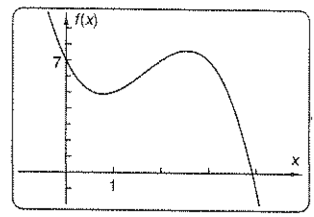
c. There is no inflection point at $x = 0$ because concavity is down for both sides, but there is an inflection point at $x = 1$.

32. a. $f(x) = x^{1.2} - 3x^{0.2}$
$f'(x) = 1.2x^{0.2} - 0.6x^{-0.8} = 0.6x^{-0.8}(2x-1)$
$f'(x) = 0 \Leftrightarrow x = 0.5$, and $f'(x)$ is undefined at $x = 0$ (critical points for $f(x)$).
$f''(x) = 0.24x^{-0.8} + 0.48x^{-1.8} = 0.24x^{-1.8}(x+2)$
$f''(x) = 0 \Leftrightarrow x = -2$ (critical point for $f'(x)$; $f'(0)$ is undefined, so f' has no critical point at $x = 0$).

b. $f(0) = 0^{1.2} - 3 \cdot 0^{0.2} = 0$ has only one value.

c. Curved concave up because $f''(x) > 0$ for $x < -2$

33. a. $f(x) = -x^3 + 5x^2 - 6x + 7$



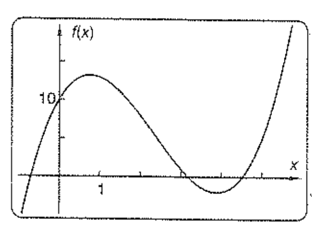
Maximum (2.5, 7.6), minimum (0.8, 4.9), points of inflection (1.7, 6.3)
No global maximum or minimum

b. $f'(x) = -3x^2 + 10x - 6$
$f'(x) = 0 \Leftrightarrow x = \frac{1}{3}(5 \pm \sqrt{7}) = 2.5485...$ or 0.7847...
$f''(x) = -6x + 10;\ f''(x) = 0 \Leftrightarrow x = \frac{5}{3} =$ 1.666...

c. $f''(0.7847...) = -6(0.7847...) + 10 =$ 5.2915... > 0, confirming local minimum.

d. Critical and inflection points occur only where f, f', or f'' is undefined (no such points exist) or is zero (all such points are found above).

34. a. $f(x) = x^3 - 7x^2 + 9x + 10$



Maximum (0.8, 13.2), minimum (3.9, −2.1), points of inflection (2.3, 5.6)
No global maximum or minimum

b. $f'(x) = 3x^2 - 14x + 9$

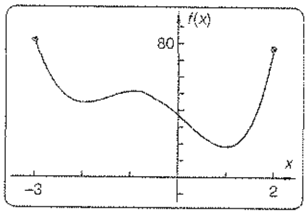
$f'(x) = 0$ at $x = \frac{1}{3}(7 \pm \sqrt{22}) = 3.896\ldots$ or $0.769\ldots$

$f''(x) = 6x - 14$; $f''(x) = 0$ at $x = \frac{7}{3} = 2.333\ldots$

c. $f''(0.769\ldots) = 6(0.769\ldots) - 14 = -9.3808\ldots < 0$, confirming local maximum.

d. Critical and inflection points occur only where f, f', or f'' is undefined (no such points exist) or is zero (all such points are found above).

35. a. $f(x) = 3x^4 + 8x^3 - 6x^2 - 24x + 37$, $x \in [-3, 2]$



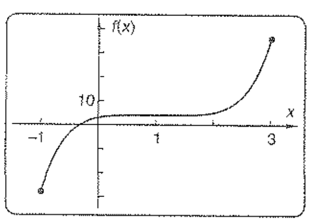
Maximum $(-3, 82)$, $(-1, 50)$, $(2, 77)$, minimum $(-2, 45)$, $(1, 18)$, points of inflection $(-1.5, 45.7)$, $(0.2, 32.0)$
Global maximum at $(-3, 82)$ and global minimum at $(1, 18)$

b. $f'(x) = 12x^3 + 24x^2 - 12x - 24$
$= 12(x + 2)(x - 1)(x + 1)$
$f'(x) = 0 \Leftrightarrow x = -2, -1, 1$
$f'(x)$ is undefined $\Leftrightarrow x = -3, 2$.
$f''(x) = 36x^2 + 48x - 12 = 12(3x^2 + 4x - 1)$;
$f''(x) = 0 \Leftrightarrow x = -\frac{1}{3}(2 \pm \sqrt{7}) = 0.2152\ldots$ or $-1.5485\ldots$
$f''(x)$ is undefined $\Leftrightarrow x = -3, 2$.

c. $f''(-2) = 12[3(4) + 4(-2) - 1] = 36 > 0$, confirming local minimum.

d. Critical and inflection points occur only where f, f', or f'' is undefined (only at endpoints) or is zero (all such points are found above).

36. a. $f(x) = (x - 1)^5 + 4$, $x \in [-1, 3]$



Maximum $(3, 36)$, minimum $(-1, -28)$, plateau and points of inflection $(1, 4)$
Global maximum at $(3, 36)$ and global minimum at $(-1, -28)$

b. $f'(x) = 5(x - 1)^4$
$f'(x) = 0 \Leftrightarrow x = 1$; $f'(x)$ is undefined $\Leftrightarrow$ $x = -1, 3$.
$f''(x) = 20(x - 1)^3$;
$f''(x) = 0 \Leftrightarrow x = 1$; $f''(x)$ is undefined $\Leftrightarrow$ $x = -1, 3$.

c. $f''(1) = 20(1 - 1)^3 = 0$, so the test fails.

d. Critical and inflection points occur only where f, f', or f'' is undefined (only at endpoints) or is zero (all such points are found above).

37. $f(x) = ax^3 + bx^2 + cx + d$; $f'(x) = 3ax^2 + 2bx + c$;
$f''(x) = 6ax + 2b \Rightarrow f''(x) = 0$ at $x = -b/(3a)$
Because the equation for $f''(x)$ is a line with nonzero slope, $f''(x)$ changes sign at $x = -b/(3a)$, so there is a point of inflection at $x = -b/(3a)$.

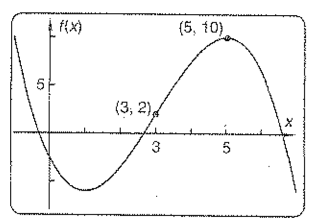
38. $f(x)$ may not have a local maximum or minimum (if $f'(x)$ is never zero); if this is not the case, then the maximum and minimum occur where $f'(x) = 3ax^2 + 2bx + c = 0$, at

$$x = \frac{-2b \pm \sqrt{4b^2 - 4 \cdot 3a \cdot c}}{6a} = \frac{-b}{3a} \pm \frac{\sqrt{b^2 - 3ac}}{3a},$$

and the maximum and minimum occur at $\sqrt{b^2 - 3ac}/(3a)$ units on either side of the inflection point $-b/(3a)$ (see Problem 33).

39. $f(x) = ax^3 + bx^2 + cx + d$
$f'(x) = 3ax^2 + 2bx + c$; $f''(x) = 6ax + 2b$
Points of inflection at $(2, 3) \Rightarrow f''(3) = 0 \Rightarrow$ $18a + 2b = 0$
Maximum at $(5, 10) \Rightarrow f'(5) = 0 \Rightarrow 75a + 10b + c = 0$
$(3, 2)$ and $(5, 10)$ are on the graph $\Rightarrow$
$27a + 9b + 3c + d = 2$.
$125a + 25b + 5c + d = 10$
Solving this system of equations yields

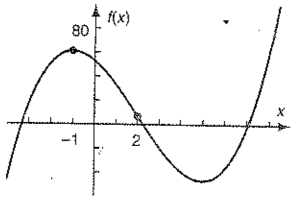
$$f(x) = -\frac{1}{2}x^3 + \frac{9}{2}x^2 - \frac{15}{2}x - \frac{5}{2}.$$



The graph confirms maximum $(5, 10)$ and points of inflection $(3, 2)$.

40. $f(x) = ax^3 + bx^2 + cx + d$
$f'(x) = 3ax^2 + 2bx + c$; $f''(x) = 6ax + 2b$
Points of inflection at $(2, 7) \Rightarrow f''(2) = 0 \Rightarrow$ $12a + 2b = 0$
Maximum at $(-1, 61) \Rightarrow f'(-1) = 0 \Rightarrow$ $3a - 2b + c = 0$

$(2, 7)$ and $(-1, 61)$ are on the graph $\Rightarrow$
$8a + 4b + 2c + d = 7$.
$-a + b - c + d = 61$
Solving this system of equations yields
$f(x) = x^3 - 6x^2 - 15x + 53$.



The graph confirms maximum $(-1, 61)$ and points of inflection $(2, 7)$.

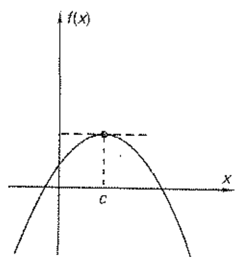
41. a. $f(x) = x^3 \Rightarrow f'(x) = 3x^2$
$f'(-0.8) = 1.92$
$f'(-0.5) = 0.75$
$f'(0.5) = 0.75$
$f'(0.8) = 1.92$

b. The slope seems to be decreasing from -0.8 to -0.5; $f''(x) = 6x < 0$ on $-0.8 \le x \le -0.5$, which confirms that the slope decreases. The slope seems to be increasing from 0.5 to 0.8; $f''(x) = 6x > 0$ on $0.5 \le x \le 0.8$, which confirms that the slope increases.

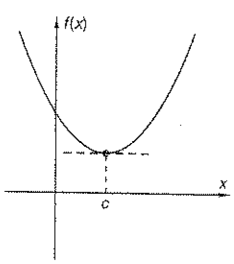
c. The curve lies above the tangent line.

42. Ima could notice that $y' = 0$ at $x = 0$ (or $y' = 3$ at $x = \pm 1$), so the graph could not possibly be a straight line with slope = 1.

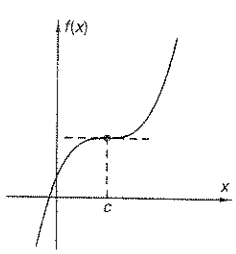
43. a.



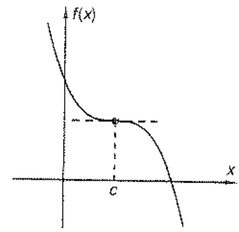
b.



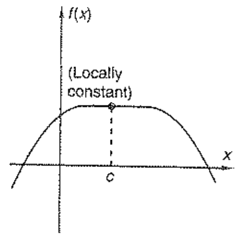
c.



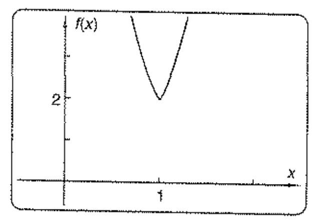
d.



e.



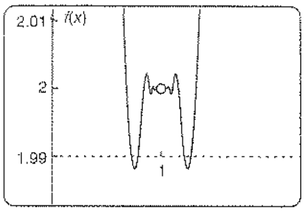
44. $f(x) = 10(x - 1)^{4/3} + 2$
$f(1) = 2$, so $f(1)$ is defined.
$f'(x) = \dfrac{40}{3}(x-1)^{1/3}$
$f'(1) = 0$, so f is differentiable at $x = 1$.
$f''(x) = \dfrac{40}{9}(x-1)^{-2/3}$
$f''(1)$ has the form $\dfrac{40}{9}(0^{-2/3})$ or $\dfrac{40}{9}(1/0)$, so $f''(1)$ is infinite.
There seems to be a cusp at $(1, 2)$, but zooming in on this point reveals that the tangent is actually horizontal there.



See Problem 20 in Problem Set 10-6 for calculation of curvature.

45. $f(x) = e^{0.06x}$, $f'(x) = 0.06e^{0.06x}$,
$f''(x) = 0.0036e^{0.06x}$
$g(x) = 1 + 0.06x + 0.0018x^2 + 0.000036x^3$
$g'(x) = 0.06 + 0.0036x + 0.000108x^2$
$g''(x) = 0.0036 + 0.000216x$
$f(0) = 1$ and $g(0) = 1$
$f'(0) = 0.06$ and $g'(0) = 0.06$
$f''(0) = 0.0036$ and $g''(0) = 0.0036$
(In fact, $f'''(0) = g'''(0)$.)
But $f(10) = e^{0.6} = 1.822\ldots \ne g(10) = 1.816$;
$f'(10) = 0.109\ldots \ne g'(10) = 0.1068$.
Because $f(x) > 0$ for all x, f has no x-intercept.
But $g(0) = 1$ and $g(-100) = -23$.
By the intermediate value theorem, $g(x) = 0$ somewhere between $x = -100$ and $x = 0$, meaning that g does have an x-intercept.

46. $f(x)=\begin{cases}(x-1)^3\sin\dfrac{1}{x-1}+2, \text{ if } x\neq 1\\ 2, \text{ if } x=1\end{cases}$



$\lim_{x\to 1} f(x)=\lim_{x\to 1}(x-1)^3\cdot\sin\dfrac{1}{x-1}+2=2=f(1)$

(The limit of the first term is zero because $(x-1)^3$ approaches zero and the sine factor is bounded.)

$\therefore f$ is continuous at $x=1$.

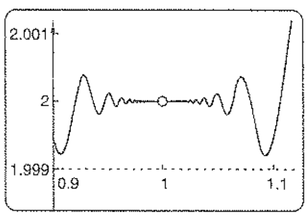
$$f'(1)=\lim_{x\to 1}\frac{f(x)-f(1)}{x-1}$$

$$=\lim_{x\to 1}\frac{[(x-1)^3\sin(1/(x-1))]+2-2}{x-1}$$

$$=\lim_{x\to 1}(x-1)^2\sin\frac{1}{x-1}=0$$

$(x-1)^2\to 0$ and the sine factor is bounded.

$\therefore f'(1)=0$



The graph is zoomed in by a factor of 10 both ways. The graph does appear to be locally linear at $x=2$. Although the sine factor makes an infinite number of cycles in any neighborhood of $x=1$, the $(x-1)^3$ factor approaches zero so rapidly that the graph is "flattened out." The name *pathological* is used to describe the fact that the graph makes an infinite number of cycles in a bounded neighborhood of $x=1$.

47. Answers will vary.

Problem Set 8-3

Q1. $y'=-3(3x+5)^{-2}$

Q2. $\ln|x+6|+C$

Q3. $-\dfrac{2}{3}x^{-5/3}$

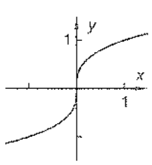
Q4. $3x^{1/3}+C$

Q5. $-x^{-1}+C$

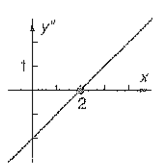
Q6. $x+C$

Q7. $\ln|\sin x|+C$

Q8.

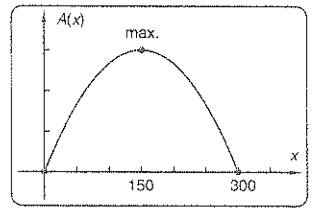


Q9.



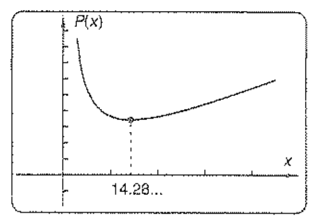
Q10. D

1. Let x = total width of pen, y = length of pen.
   Domains: $0\le x\le 300$, $0\le y\le 200$
   Maximize $A(x)=xy$.
   $2x+3y=600\Rightarrow y=200-\dfrac{2}{3}x$
   $\therefore A(x)=200x-\dfrac{2}{3}x^2$



   The graph shows a maximum at $x\approx 150$.
   Algebraically, $A'(x)=200-\dfrac{4}{3}x$.
   $A'(x)=0\Leftrightarrow x=150$, confirming the graph.
   $x=150\Rightarrow y=200-\dfrac{2}{3}\cdot 150=100$
   Make the total width 150 ft and length 100 ft.
   (*Note:* The maximum area was not asked for.)

2. a. Let x = width of a room across the front, y = depth of a room from front to back.
   Domains: $x\ge 0$, $y\ge 0$
   Minimize $P(x)=12x+7y$.
   $xy=350\Rightarrow y=350x^{-1}$
   $\therefore P(x)=12x+2450x^{-1}$



   The graph shows a minimum at $x\approx 14$.
   Algebraically, $P'(x)=12-2450x^{-2}$.
   $P'(x)=0\Leftrightarrow 2450x^{-2}=12\Leftrightarrow$
   $x=\pm\sqrt{2450/12}=\pm 35/\sqrt{6}=\pm 14.288\ldots$
   Minimum is at $x=35/\sqrt{6}$, $y=10\sqrt{6}=24.49\ldots$.
   Make rooms 14.3 ft across and 24.5 ft deep.

   b. For 10 rooms, $P(x)=20x+11y=20x+3850x^{-1}$.
   $P'(x)=20-3850x^{-2}=0$ at $x=\sqrt{192.5}$
   Minimum at $x=\sqrt{192.5}=13.87\ldots$,
   $y=350/\sqrt{192.5}=25.22\ldots$
   Make rooms 13.9 ft across and 25.2 ft deep.
   For 3 rooms, $P(x)=6x+4y=6x+1400x^{-1}$.
   $P'(x)=6-1400x^{-2}=0$ at $x=\sqrt{1400/6}=10\sqrt{7/3}$

Minimum at $x = 10\sqrt{7/3} = 15.27\ldots$,

$y = 5\sqrt{21} = 22.91\ldots$

Make rooms 15.3 ft across and 22.9 ft deep.

3. a. Let x = width of rectangle, $2x$ = length of rectangle, y = width of square.

$A_{rect} = 2x^2$, $A_{sq} = y^2$

For minimal rectangle, $2x^2 \ge 800 \Rightarrow x \ge 20$.

For minimal square, $y^2 \ge 100 \Rightarrow y \ge 10$.

Perimeter $P = 6x + 4y = 600 \Rightarrow$

$y = 150 - 1.5x$

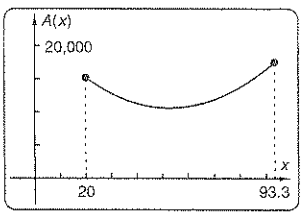
$\therefore\ 150 - 1.5x \ge 10 \Rightarrow x \le 140/1.5 = 93.3333\ldots$

Domain: $20 \le x \le 93.3333\ldots$

b. Total area $A(x) = 2x^2 + y^2$

$= 2x^2 + (150 - 1.5x)^2$

$= 22500 - 450x + 4.25x^2$



c. The graph shows a maximum at endpoint $x = 93.3333\ldots$.

$A'(x) = -450 + 8.5x$

$A'(x) = 0 \Leftrightarrow x = 450/8.5 = 52.9411\ldots$

Because $A(52.9\ldots)$ is a *minimum*, the maximum occurs at an endpoint.

$A(20) = 15200$, $A(93.3333\ldots) = 17522.2222\ldots$

Greatest area $\approx$ 17,522 ft^2

4. a. Let r = radius of circle, s = width of square

Diameter $\ge 50 \Rightarrow r \ge 25$

Circumference $\le 1000 \Rightarrow 2\pi r \le 1000 \Rightarrow$

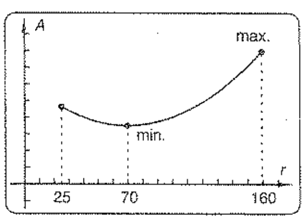
$r \le 500/\pi$

Domain of r: $25 \le r \le 500/\pi = 159.154\ldots$

Minimize $A(r) = \pi r^2 + s^2$.

Perimeter $2\pi r + 4s = 1000 \Rightarrow s = 250 - \pi r/2$

$\therefore\ A(r) = \pi r^2 + (250 - \pi r/2)^2$



The graph shows minimum area at $x \approx 70$.

$A'(r) = 2\pi r + 2(250 - \pi r/2)(-\pi/2)$

$A'(r) = 0 \Leftrightarrow 2\pi r - \pi(250 - \pi r/2) = 0 \Rightarrow$

$r = 500/(4 + \pi) = 70.012\ldots$

$A(25) = 46370.667\ldots$

$A(70.012\ldots) = 35006.197\ldots$

$A(159.154\ldots) = 79577.471\ldots$

Minimum area at $r = 70.012\ldots$,

$s = 1000/(4 + \pi) = 140.024\ldots$

For square, $4(140.024\ldots) \approx 560$.

For circle, $2\pi(70.012\ldots) \approx 440$.

Use 440 yd for square and 560 yd for circle. (You could build a square corral with side 140 *around* the circular fence of radius 70 to enclose a total area of only 19,607 yd^2, but Big Bill might not like your solution!)

b. The graph of A versus r shows that the maximum area occurs at the largest possible circle. Big Bill should use all 1000 yards for the circular fence and not build a corral.

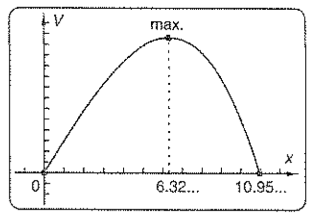
5. a. Let x = length of square base, z = height of box.

Domain of x: $0 \le x \le \sqrt{120} = 10.954\ldots$

Maximize $V(x) = x^2 z$.

Area $= x^2 + 4xz = 120 \Rightarrow z = 30/x - x/4$

$\therefore\ V(x) = 30x - x^3/4$



The graph shows a maximum at $x \approx 6.3$.

$V'(x) = 30 - 3x^2/4 = 0$ at $x = \pm\sqrt{40}$

$x = -\sqrt{40}$ is out of the domain.

Critical points at $x = 0, x = \sqrt{40}, x = \sqrt{120}$

$V(0) = 0,\ V(\sqrt{120}) = 0$

$V(\sqrt{40}) = 20\sqrt{40} = 126.49\ldots$

Maximum at $x = \sqrt{40} = 6.324\ldots$,

$z = \sqrt{40}/2 = 3.162\ldots$

Make the box 6.32 cm square by 3.16 cm deep.

b. Conjecture: An open box with square base of side length x and fixed surface area A will have maximal volume when the base length is twice the height, which occurs when $x = \sqrt{A/3}$ (see the solution to Problem 8b).

6. a. Domain of x is $0 \le x \le 6$.

b. $V(0) = 0$ cm^2

$V(1) = 180$ cm^2

$V(2) = 256$ cm^2 (largest volume for an integer value of x)

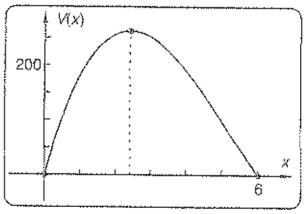
$V(3) = 252$ cm^2

$V(4) = 192$ cm^2

$V(5) = 100$ cm^2

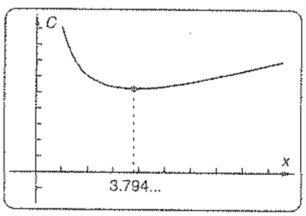
$V(6) = 0$ cm^2

c. $V(x) = (20 - 2x)(12 - 2x)x$
$= 240x - 64x^2 + 4x^3$



The graph shows a maximum at $x \approx 2.4$.
$V'(x) = 240 - 128x + 12x^2 = 0$ at
$x = (128 \pm \sqrt{4864})/24 = 2.427\ldots$ or $8.239\ldots$
$x = 8.239\ldots$ is out of the domain.
$V(2.427\ldots) = 262.68\ldots$ is a maximum because it is positive and $V(0) = V(6) = 0$.
Maximum volume ≈ 262.7 cm^2 at $x \approx 2.43$ cm

7. Let x = length, y = depth, $C(x)$ = total cost.
Domains: $x > 0$, $y > 0$
Area of bottom = $5x$
Total area of sides is $(10 + 2x)y$.
Minimize $C(x) = 10(5x) + 5(10 + 2x)y$.
Volume $= 72 \Rightarrow 5xy = 72 \Rightarrow$
$y = 72/(5x) = 14.4x^{-1}$
$\therefore C(x) = 50x + 5(10 + 2x)(14.4x^{-1})$
$C(x) = 50x + 720x^{-1} + 144$



The graph shows a minimum at $x \approx 3.8$.
$C'(x) = 50 - 720x^{-2} = 0 \Leftrightarrow x = \pm\sqrt{72/5} = \pm 3.7947\ldots$
$x = -3.7947\ldots$ is out of the domain.
Minimum is at $x = 3.7947$ because $C'(x)$ changes from negative to positive there.
$C(3.7947\ldots) = 120\sqrt{10} + 144 \approx 523.47$
Minimum cost is \$523.47.

8. a. Maximize $V = xyz$.
Fixed area $A = xy + 2xz + 2yz$
$\Rightarrow y = (A - 2xz)/(x + 2z)$

$$\therefore V = \frac{Axz - 2x^2z^2}{x + 2z}$$

$$\frac{dV}{dx} = \frac{-2z^2x^2 - 8z^3x + 2Az^2}{(x + 2z)^2}$$

$$\frac{dV}{dx} = 0 \text{ at } x = -2z + \sqrt{4z^2 + A}$$

$$y = \frac{A - 2z(-2z + \sqrt{4z^2 + A})}{-2z + \sqrt{4z^2 + A} + 2z}$$

$$= -2z + \sqrt{4z^2 + A}$$

Therefore, $x = y$ for maximum volume, Q.E.D.

b. Let $x = y$. Maximize $V = xyz = x^2z$.
Fixed area $A = xy + 2xz + 2yz = x^2 + 4xz$
$\Rightarrow z = A/(4x) - x/4$
$\therefore V = (A/4)x - x^3/4$

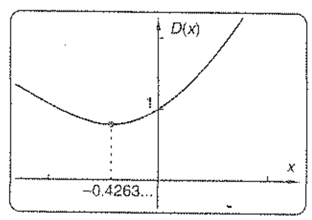
$$\frac{dV}{dx} = (A/4) - 3x^2/4 = 0 \text{ at } x = \pm\sqrt{A/3}$$

dV/dx goes from positive to negative at $x = \sqrt{A/3} \Rightarrow$ maximum at $x = \sqrt{A/3}$.

$$z = A/(4\sqrt{A/3}) - \sqrt{A/3}/4 = \frac{1}{2}\sqrt{A/3} = \frac{1}{2}x$$

c. For the maximal box in part b, the depth is half the length of the base. Thus, the box is short and fat. This makes sense because the problem is equivalent to maximizing the volume of two open boxes with the second box placed upside-down on the first. The resulting single closed box will have maximum volume when it is a cube, which will happen if each open box is half a cube.

9. For $y = e^x$, minimize $D(x) = \sqrt{x^2 + y^2} = \sqrt{x^2 + e^{2x}}$.

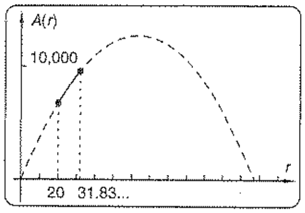


The graph shows a minimum at $x \approx -0.43$.

$$D'(x) = \frac{1}{2}(x^2 + e^{2x})^{-1/2}(2x + 2e^{2x})$$

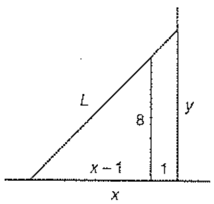
$D'(x) = 0 \Leftrightarrow 2x + 2e^{2x} = 0 \Leftrightarrow x = -e^{2x}$
Because x appears both algebraically and exponentially, there is no analytic solution. Solving numerically gives $x \approx -0.4263$. By graphing $D(x)$, $D(-0.4263)$ is a minimum.
Closest point to the origin is $(x, y) = (-0.4263\ldots, 0.6529\ldots)$.

10. Minimize $A(r) = \pi r^2 + 2rx$, $r \geq 20$.
$2\pi r + 2x = 400 \Rightarrow x = 200 - \pi r$
$x \geq 100 \Rightarrow r \leq 100/\pi$
$\therefore$ domain is $20 \leq r \leq 100/\pi$.
$A(r) = \pi r^2 + 2r(200 - \pi r) = 400r - \pi r^2$



The graph shows a minimum at endpoint $x = 20$.
$A' = 400 - 2\pi r$
$A' = 0 \Leftrightarrow r = 200/\pi = 63.6\ldots$ (out of domain)
$A' > 0$ for all r in the domain.
$\therefore$ minimum occurs at left end of domain, $r = 20$.
$x = 200 - 20\pi = 137.168\ldots$
Make radius of semicircles 20 m and straight sections 137.17 m.

11.

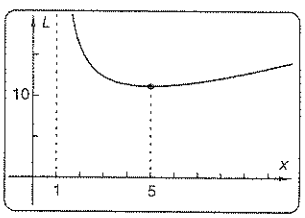


$L(x) = \sqrt{x^2 + y^2}$.

Domains: $x \geq 1$, $y \geq 8$
Minimize $L^2(x) = x^2 + y^2$.

Using similar triangles, $\dfrac{y}{x} = \dfrac{8}{x-1} \Rightarrow y = \dfrac{8x}{x-1}$.

$\therefore L^2(x) = x^2 + \dfrac{64x^2}{(x-1)^2}$



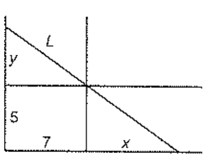
The graph shows a minimum of $L(x)$ at $x \approx 5$.

$(L^2)'(x) = 2x - \dfrac{128x}{(x-1)^3}$

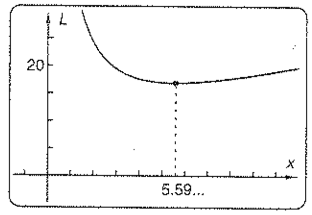
$(L^2)'(x) = 0 \Leftrightarrow 2x = \dfrac{128x}{(x-1)^3} \Leftrightarrow$

$x = 0$ (out of domain) or $(x - 1)^3 = 64 \Leftrightarrow x = 5$
By graph, $L(x)$ is a minimum at $x = 5$.
Shortest ladder has length $L(5) = 5\sqrt{5} \approx 11.18$ ft.

12. Let x and y be the segments shown.

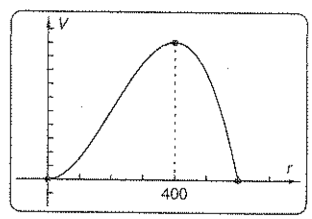


$L(x) = \sqrt{(x+7)^2 + (y+5)^2}$
Maximize $L^2(x) = (x + 7)^2 + (y + 5)^2$.
Using similar triangles, $y/7 = 5/x \Rightarrow y = 35/x$.
$\therefore L^2(x) = (x + 7)^2 + (35/x + 5)^2$
$L^2(x) = x^2 + 14x + 49 + 1225/x^2 + 350/x + 25$



The graph shows a minimum of $L(x)$ at $x \approx 5.6$.
$(L^2(x))' = 2x + 14 - 350x^{-2} - 2450x^{-3}$
By numerical solution, $(L^2)' = 0$ at $x \approx 5.5934\ldots$.
(Exact answer is $x = \sqrt[3]{175}$.)
But a minimum distance L in the hall implies that the *maximal* ladder that will go through the hall is at $x = 5.5934\ldots$.
$L^2(5.5934\ldots) = 285.3222\ldots$
$L(5.5934\ldots) = 16.8914\ldots$
No ladder longer than 16.8 ft (rounded down) can pass through the hall.

13. Let r = radius, h = height.
$V = \pi r^2 h$
$2r + 2h = 1200 \Rightarrow h = 600 - r$
$\therefore V = \pi r^2(600 - r) = \pi(600r^2 - r^3)$



The graph shows a maximum at $r \approx 400$.
$V' = \pi(1200r - 3r^2)$
$V' = 0 \Leftrightarrow r = 0$ or $r = 400$
From graph, maximum is at $r = 400$.
$h = 600 - 400 = 200$
Maximum volume occurs with rectangle 400 mm wide (radius), 200 mm high.

14. Rotating a square does *not* give the maximum volume. The solution to Problem 13 gives a counterexample. Repeating the calculations with perimeter P instead of 1200 gives $r = (1/3)P$ and $h = (1/6)P$, showing that the proportions for maximum volume are with radius *twice* the height.

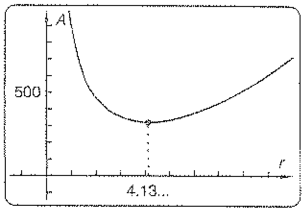
15. a. Let r = radius, h = height.
$V = \pi r^2 h = \pi(3.65^2)(10.6) = 141.2185\pi$
$= 443.6510\ldots$ cm^3

b. $A = 2\pi rh + 2\pi r^2$

$V = \pi r^2 h = 141.2185\pi \Rightarrow h = 141.2185/r^2$

$\therefore A = 2\pi r(141.2185/r^2) + 2\pi r^2$

$A = 2\pi(141.2185r^{-1} + r^2)$

c. 

The graph shows a minimum at $x \approx 4.1$.

$A' = 2\pi(-141.2185r^{-2} + 2r)$

$A' = 2\pi/r^2(-141.2185 + 2r^3)$

$A' = 0 \Leftrightarrow r^3 = 70.60925 \Rightarrow$

$r = \sqrt[3]{70.60925} = 4.1332\ldots$

Minimum at $r = 4.1\ldots$ because A' goes from negative to positive.

$h = 141.2815/(\sqrt[3]{70.60925})^2 = 2\sqrt[3]{70.60925}$
$= 8.2664\ldots$

Radius ≈ 4.1 cm, height ≈ 8.3 cm

Because height = 2 × radius, height = diameter. So minimal can is neither tall and narrow nor short and wide.

d. Normally proportioned can is taller and narrower than minimal can. For normal can, $A = 2\pi(3.65)(10.6) + 2\pi(3.65)^2 =$ 326.8041... .

For minimal can, $A = 2\pi(4.13\ldots)(8.26\ldots) + 2\pi(4.13\ldots)^2 = 322.014\ldots$.

Difference is 4.78... cm^2.

Percent: (4.789...)(100)/326.80... = 1.465...
≈ 1.5% of metal in normal can

e. Savings = $(0.06)(20 \times 10^6)(0.01465\ldots)(365) = 6.419\ldots \times 10^6$, or about \$6.4 million!

16. a. $C(r) = 2\pi r^2 k + 2\pi rh = 2\pi r^2 k + 282.437\pi r^{-1}$

$C'(r) = 4\pi rk - 282.437\pi r^{-2}$
$= 4\pi r^{-2}(kr^3 - 70.60925)$

$C'(r) = 0$ at $r = \sqrt[3]{70.60925/k}$

$C''(r) = 4\pi k + 564.874\pi r^{-3} > 0$ for all $r > 0$, so this is a local minimum.

If the normal can is the cheapest to make, then $3.65 = \sqrt[3]{70.60925/k} \Rightarrow$

$k = 70.60925(3.65)^{-3} = 1.4520\ldots$.

This is reasonable because metal for the ends is cut into circles, so some must be wasted.

b. Now it takes $(2r)^2$ cm^2 of metal to make each end of the can, so the function to minimize is $C(r) = 8r^2k + 2\pi rh = 8r^2k + 282.437\pi r^{-1}$.

$C'(r) = 16rk - 282.437\pi r^{-2}$

$C'(r) = 0$ at $r = \sqrt[3]{\dfrac{282.437\pi}{16k}}$

$C''(r) = 16k + 564.874\pi r^{-3} > 0$ for all $r > 0$, so this is a local minimum.

If the normal can is the cheapest to make, then $3.65 = \sqrt[3]{\dfrac{282.437\pi}{16k}} \Rightarrow k = \dfrac{282.437\pi}{16(3.65)^3}$
$= 1.1404\ldots$.

To minimize the area (not the cost) of the can, minimize $8r^2 + 2\pi rh = 8r^2 + 282.437\pi r^{-1}$.

$C'(r) = 16r - 282.437\pi r^{-2} = 0 \Rightarrow$

$r = \sqrt[3]{\dfrac{282.437\pi}{16}} = 3.8126$ cm

$h = \dfrac{141.2185}{\left(\sqrt[3]{282.437\pi/16}\right)^2} = 9.7099\ldots$ cm.

The proportions of this can are closer to those of the normal can.

c. If the metal for the ends can be cut without waste, then it takes $\pi(r + 0.6)^2$ to make each end and $(2\pi r + 0.5)h$ to make the sides, so minimize

$C(r) = 2\pi(r + 0.6)^2 + (2\pi r + 0.5)h$
$= 2\pi(r+0.6)^2 + 141.2185(2\pi r + 0.5)r^{-2}$

$C'(r) = 4\pi(r+0.6) - 282.437\pi r^{-2}$
$- 141.2185r^{-3}$

$C'(r) = 0$ at $r \approx 3.9966$ by graphing calculator.

$C''(r) = 4\pi + 564.874\pi r^{-3} + 423.6555r^{-4} > 0$ for all $r > 0$, so this is a minimum point.

Minimal can has $r \approx 3.9966\ldots$,
$h \approx 8.8411\ldots$ cm.

But if the metal for the ends is cut from squares, then it takes $4(r + 0.6)^2$ to make each end and $(2\pi r + 0.5)h$ to make the sides, so minimize:

$C(r) = 8(r + 0.6)^2 + (2\pi r + 0.5)h$
$= 8(r+0.6)^2 + 141.2185(2\pi r + 0.5)r^{-2}$

$C'(r) = 16(r+0.6) - 282.437\pi r^{-2}$
$- 141.2185r^{-3}$

$C'(r) = 0$ at $r \approx 3.6776\ldots$ by graphing calculator.

$C''(r) = 16 + 564.874\pi r^{-3} + 423.6555r^{-4} > 0$ for all $r > 0$, so this is a minimum point.

Minimal can has $r \approx 3.6776\ldots$,
$h \approx 10.4411\ldots$.

This is close to the normal can!

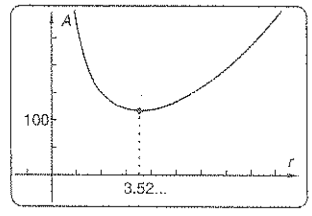
17. a. Volume of cup = $\pi(2.5)^2 \cdot 7 = 43.75\pi$

Let r = radius of cup, h = height of cup.

Minimize $A(r) = \pi r^2 + 2\pi rh$.

$\pi r^2 h = 43.75\pi \Rightarrow h = 43.75r^{-2}$

$\therefore A(r) = \pi r^2 + 87.5\pi r^{-1}$



The graph shows a minimum at $r \approx 3.5$ cm.
$A'(r) = 2\pi r - 87.5\pi r^{-2} = 2\pi r^{-2}(r^3 - 43.75)$
$A'(r) = 0$ at $r = \sqrt[3]{43.75} = 3.5236\ldots$.
There is a minimum at $x = 3.5236\ldots$ because $A(r)$ goes from decreasing to increasing. (See graph.)
$h = 43.75(43.75)^{-2/3} = \sqrt[3]{43.75} = r$
Minimal cup has $r \approx 3.52$ cm, $h \approx 3.52$ cm.

b. Ratio is $d : h = 2r : h = 2 : 1$.

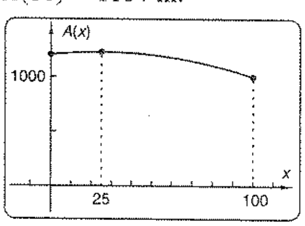
c. Current cup design uses $\pi(2.5)^2 + \pi \cdot 5 \cdot 7 = 41.25\pi = 129.59\ldots \text{ cm}^2 = 0.012959\ldots \text{ m}^2$ per cup, which costs
$(300{,}000{,}000)(0.012959\ldots)(2.00)$
$\approx \$7{,}775{,}441.82$ per year.
Minimal cup design uses $3\pi(43.75)^{2/3} = 117.01\ldots \text{ cm}^2 = 0.011701\ldots \text{ m}^2$ per cup, which costs $(300{,}000{,}000)(0.011701\ldots)(2.00)$
$\approx \$7{,}021{,}141.88$ per year.
Switching to minimal cup design would save $754{,}299.93 \approx \$754{,}000$ per year in paper costs (about 10% of the current annual paper bill), but would likely result in loss of sales because a cup of that shape is hard to drink from.

d. Let r = radius of cup, h = height of cup.
$\pi r^2 h = V \Rightarrow h = (V/\pi)r^{-2}$
Minimize $A(r) = \pi r^2 + 2\pi rh = \pi r^2 + 2Vr^{-1}$.
$A'(r) = 2\pi r - 2Vr^{-2} = 0$ at $r = \sqrt[3]{V/\pi}$
$A''(r) = 2\pi + 4Vr^{-3} > 0$ for all $r > 0$, so this is a minimum.
Minimal cup has $r = \sqrt[3]{V/\pi}$,
$h = (V/\pi)(V/\pi)^{-2/3} = \sqrt[3]{V/\pi} = r$.

18. a. $A = yz = (30 + 0.2x)(40 - 0.2x)$
$A(x) = 1200 + 2x - 0.04x^2$
Left rectangle: $A(0) = 1200$ in.^{2}
Right rectangle: $A(100) = 1000$ in.^{2}

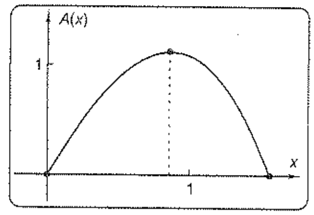
b. $A(80) = 1104$ in.^{2}

c.



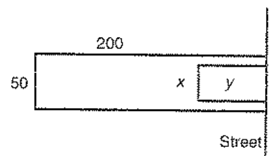
The graph shows a maximum at $x \approx 25$.
$A'(x) = 2 - 0.08x = 0$ at $x = 25$.
Critical points at $x = 0, 25, 100$
$A(25) = 1225$ in.^{2}; $A(0) = 1200$ in.^{2};
$A(100) = 1000$ in.^{2} (from part a)
Maximum area at $x = 25$ in., minimum area for $x = 100$ in.

19. Maximize $A(x) = 2xy = 2x \cos x$.
Use $0 \le x \le \pi/2$ for the domain of x.

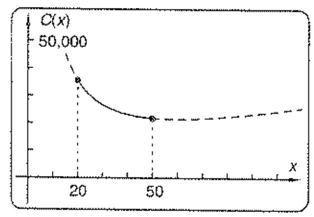


The graph shows a maximum at $x \approx 0.86$.
$A'(x) = 2 \cos x - 2x \sin x$
$A'(x) = 0$ when $x = \cot x$.
Solving numerically gives $x \approx 0.8603\ldots$.
$A(0) = A(\pi/2) = 0$; $A(0.8603\ldots) = 1.1221\ldots$
Maximum area = $1.1221\ldots$

20.



Let x = width of store, y = length of store.
Minimize $C(x) = 100x + 80(x + 2y)$.
$xy = 4000 \Rightarrow y = 4000x^{-1}$
$C(x) = 180x + 640000x^{-1}$
$y \le 200 \Rightarrow x \ge 20$, so domain of x is $20 \le x \le 50$.
Graph shows minimum at x endpoint $x = 50$.

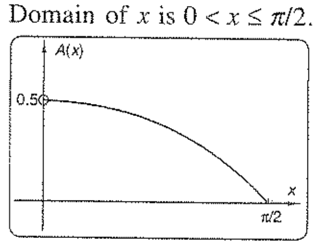


$C'(x) = 180 - 640000x^{-2} = 0$
at $x = \dfrac{80\sqrt{5}}{3} = 59.628\ldots$, outside the domain.
$C(20) = \$35{,}600.00$; $C(50) = \$21{,}800.00$
Minimum cost is at $x = 50$, $y = 4000/50 = 80$.
Bill should build the store 50 ft × 80 ft.

21. a. $A = 0.5xy = 0.5x \cot x$

$$\lim_{x\to 0} A = \lim_{x\to 0} \frac{x}{2 \tan x} \longrightarrow \frac{0}{0}$$
$$= \lim_{x\to 0} \frac{1}{2\sec^2 x} = \frac{1}{2}$$

b. Domain of x is $0 < x \le \pi/2$.



The graph shows that the area approaches a maximum as x approaches the endpoint $x = 0$ from the positive side.

$A'(x) = \frac{1}{2}(\cot x - x\csc^2 x)$

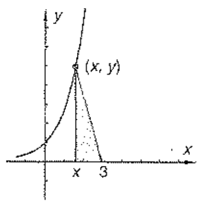
$A'(x) = 0$ when $x = \cos x \sin x$ or

$2x = 2 \sin x \cos x = \sin 2x$,

which happens at $x = 0$.

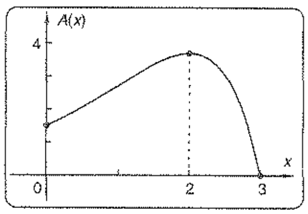
$A(\pi/2) = 0$, so the "maximum" occurs at $x = 0$. But $x = 0$ is not in the domain; $A(x)$ can get arbitrarily close to 1/2, but never achieve it.

22.



Domain of x is $0 \le x \le 3$.

Maximize $A = 0.5(3 - x)(y) = 0.5(3 - x)e^x = 1.5e^x - 0.5xe^x$.



The graph shows a maximum at $x \approx 2$.

$A'(x) = 1.5e^x - 0.5e^x - 0.5xe^x = 0.5e^x(2 - x)$

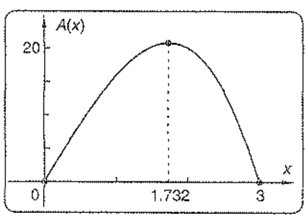
$A'(x) = 0$ at $x = 2$, confirming the graph.

$A'(x) > 0$ for $x < 2$, and $A'(x) < 0$ for $x > 2$, confirming maximum point at $x = 2$.

Maximum area $A(2) = e^2/2 = 3.69452\ldots$.

23. a. Maximize $A(x) = 2xy = 2x(9 - x^2) = 18x - 2x^3$.

Domain: $0 \le x \le 3$



The graph shows a maximum at $x \approx 1.7$.

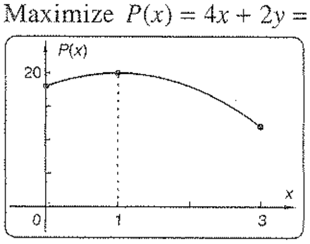
$A'(x) = 18 - 6x^2 = 0$ at $x = \pm\sqrt{3} = \pm 1.732\ldots$

-1.732 is out of the domain.

$A(0) = A(3) = 0$; $A(\sqrt{3}) = 12\sqrt{3} = 20.7846\ldots$

Maximal rectangle has width $= 2\sqrt{3}$, length $= 9 - 3 = 6$.

b. Maximize $P(x) = 4x + 2y = 4x + 18 - 2x^2$.



The graph shows a maximum at $x \approx 1$.

$P'(x) = 4 - 4x = 0$ at $x = 1$

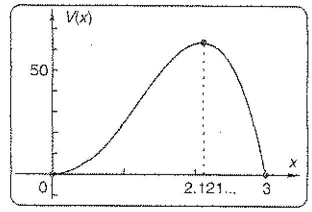
$P(0) = 18$; $P(1) = 20$; $P(3) = 12$

Maximal rectangle has width $= 2$, length $= 9 - 1 = 8$.

c. No. The maximum-area rectangle is $2\sqrt{3}$ by 6. The maximum-perimeter rectangle is 2 by 8.

24. a. Maximize $V(x) = \pi x^2 y = \pi x^2(9 - x^2) = 9\pi x^2 - \pi x^4$.

Domain: $0 \le x \le 3$



The graph shows a maximum at $x \approx 2.1$.

$V'(x) = 18\pi x - 4\pi x^3 = 0$ at $x = 0, \pm\sqrt{4.5}$.

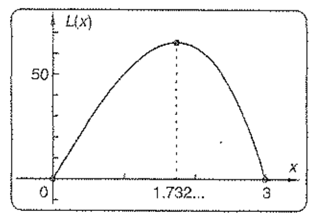
$-\sqrt{4.5}$ is out of the domain.

$V(0) = V(3) = 0$, $V(\sqrt{4.5}) = 20.25\pi = 63.6172\ldots$

Maximum is at $x = \sqrt{4.5}$, $y = 9 - 4.5 = 4.5$.

Maximal cylinder has radius $= \sqrt{4.5} = 2.12132\ldots$ and height $= 4.5$.

b. Maximize $L(x) = 2\pi xy = 2\pi x(9 - x^2) = 18\pi x - 2\pi x^3$.



The graph shows a maximum at $x \approx 1.7$.

$L'(x) = 18\pi - 6\pi x^2 = 0$ at $x = \pm\sqrt{3}$.

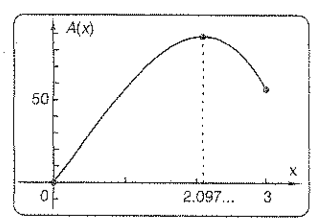
$-\sqrt{3}$ is out of the domain.

$L(0) = L(3) = 0$; $L(\sqrt{3}) = 12\pi\sqrt{3} = 65.2967\ldots$

Maximum is at $x = \sqrt{3}$, $y = 9 - 3 = 6$.

Maximal cylinder has radius $= \sqrt{3} = 1.7320\ldots$ and height $= 6$.

c. Maximize $A(x) = 2\pi x^2 + 2\pi xy = 2\pi x^2 + 2\pi x(9 - x^2) = 2\pi x^2 + 18\pi x - 2\pi x^3$.



The graph shows a maximum at $x \approx 2.1$.

$A'(x) = 18\pi + 4\pi x - 6\pi x^2$

$A'(x) = 0$ at $x = \dfrac{1 \pm 2\sqrt{7}}{3} = 2.0971\ldots$ or $-1.430\ldots$

$-1.430\ldots$ is out of the domain.

$A(0) = 0$; $A(2.0971\ldots) = 88.2727\ldots$;

$A(3) = 18\pi = 56.5486\ldots$

Maximal cylinder has radius $= \dfrac{1+2\sqrt{7}}{3} = 2.0971\ldots$ and height $= \dfrac{52-4\sqrt{7}}{9} = 4.6018\ldots$.

d. No. The maximum-volume cylinder has dimensions different from both of the maximum-area cylinders in parts b and c.

e. No. Rotating the maximum-area rectangle does not produce the maximum-volume cylinder. But it produces the cylinder with maximum lateral area.

f. If $y = a^2 - x^2$, the paraboloid has radius $= a$.

$V = \pi x^2(a^2 - x^2) = \pi(a^2x^2 - x^4)$

$V' = \pi(2a^2x - 4x^3)$

$V' = 0 \Leftrightarrow x = 0$ or $x = \pm a/\sqrt{2}$.

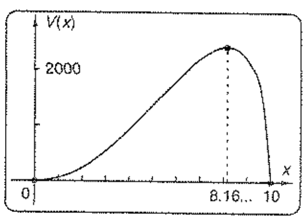
V is maximum at $x = a/\sqrt{2}$.

For the cylinder of maximum volume, (cylinder radius):(paraboloid radius) $= 1/\sqrt{2}$, a constant.

Note: This ratio is also constant ($1/\sqrt{3}$) for the cylinder of maximum lateral area, but is *not* constant for the cylinder of maximum total area.

25. a. $x^2 + y^2 = 100$, $0 \le x \le 10$

Maximize $V(x) = \pi x^2 \cdot y = 2\pi x^2\sqrt{100 - x^2}$.

b. 

The graph shows a maximum volume at $x \approx 8.2$.

$$V'(x) = \frac{-2\pi x^3}{\sqrt{100-x^2}} + 4\pi x\sqrt{100-x^2} = \frac{-6\pi x^3 + 400\pi x}{\sqrt{100-x^2}}$$

$V'(x) = 0$ at $x = 0$, $\sqrt{\dfrac{200}{3}} = \dfrac{10\sqrt{6}}{3} = 8.1649\ldots$

$V(0) = V(10) = 0$

$$V\left(\frac{10\sqrt{6}}{3}\right) = \frac{4000\pi\sqrt{3}}{9} = 2418.399\ldots$$

Maximal cylinder has radius $= 8.1649\ldots$, height $= \dfrac{20\sqrt{3}}{3} = 11.5470\ldots$, and volume $= 2418.39\ldots$.

c. Height = radius $\cdot \sqrt{2}$

Volume of sphere $V_s = \dfrac{4}{3}\pi \cdot 1000 = \dfrac{4000\pi}{3}$

Volume of maximal cylinder $V_c = \dfrac{4000\pi\sqrt{3}}{9}$

$\therefore V_c = V_s/\sqrt{3}$

26. Let r = radius of cone, h = height.

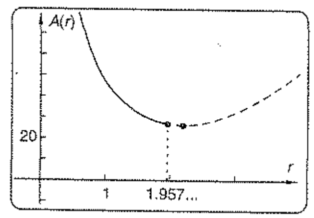
Lateral area $A(r) = \pi r \cdot$ (slant height) $= \pi r\sqrt{r^2 + h^2}$

$V = \dfrac{1}{3}\pi r^2 h = 5\pi \Rightarrow h = 15r^{-2}$

$\therefore A(r) = \pi r\sqrt{r^2 + 225r^{-4}}$

$h \ge 2r \Rightarrow 2r \le 15r^{-2}$

Domain of r is $0 < r \le \sqrt[3]{7.5} = 1.9574\ldots$.



The graph shows a minimum of $A(r)$ at endpoint $r = 1.957\ldots$.

Minimize $A^2(r) = \pi^2(r^4 + 225r^{-2})$.

$(A^2(r))' = \pi^2(4r^3 - 450r^{-3}) = 0$ at $r = \sqrt[6]{112.5} = 2.1971\ldots$, which is out of the domain.

$A(1.9574\ldots) = 26.915\ldots$, $\lim_{r \to 0^+} A(r) = \infty$.

Minimal cone has radius $= \sqrt[3]{7.5} = 1.9574\ldots$ and height $= 2r = 2\sqrt[3]{7.5} = 3.9148\ldots$.

Make $r \approx 1.96$ ft and $h \approx 3.91$ ft.

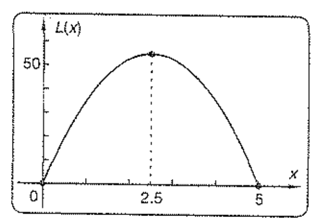
27. a. Lateral area $L(x) = 2\pi xy$

Domains: $0 \le x \le 5$ and $0 \le y \le 7$

Equation of element of cone is

$y = -\dfrac{7}{5}x + 7 \Rightarrow y = -1.4x + 7$.

$\therefore L(x) = 2\pi x(-1.4x + 7) = 2\pi(-1.4x^2 + 7x)$



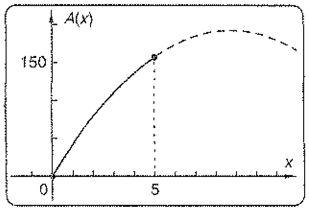
The graph shows a maximum of $L(x)$ at $x \approx 2.5$.

$L'(x) = 2\pi(-2.8x + 7)$

$L'(x) = 0$ at $x = 2.5$.

$L'(x)$ goes from positive to negative at $x = 2.5$.
$\therefore$ maximum lateral area at radius $x = 2.5$ cm.

b. Total area $A(x) = 2\pi xy + 2\pi x^2$
$= 2\pi x(-1.4x + 7) + 2\pi x^2$
$A(x) = 2\pi(7x - 0.4x^2)$



The graph shows a maximum at endpoint $x = 5$.
$A'(x) = 2\pi(7 - 0.8x) = 0$ at $x = 8.75$, out of domain.
$\therefore$ maximum is at an endpoint, $x = 5$.
$A(0) = 0$; $A(5) = 2\pi(5^2) = 50\pi = 157.07\ldots$
Maximum area is with the degenerate cylinder consisting only of the top and bottom, radius 5 and height 0.

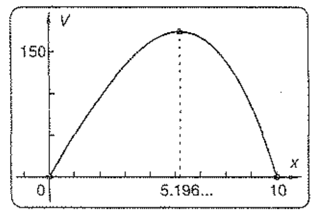
28. a. Let r = radius of cone, h = height of cone (constants).
Let (x, y) be a sample point on cone element.
Domain of x is $0 \le x \le r$.
$L(x) = 2\pi xy$.
Equation of element of cone is
$y = (-h/r)x + h$.
$\therefore L(x) = 2\pi x[(-h/r)x + h] = 2\pi h(-x^2/r + x)$
$L'(x) = 2\pi h(-2x/r + 1)$
$L'(x) = 0$ at $x = r/2$.
$L'(x)$ goes from positive to negative at $x = r/2$.
$\therefore$ maximum lateral area at radius $x = r/2$.

b. $A(x) = 2\pi xy + 2\pi x^2$
$= 2\pi x[h - (h/r)x] + 2\pi x^2$
$A(x) = 2\pi[(1 - h/r)x^2 + hx]$
$A'(x) = 2\pi[2(1 - h/r)x + h] = 0$ at
$$x = \frac{-h}{2(1 - h/r)}$$
$$A'(x) = 0 \text{ at } x = \frac{-h}{2 - 2h/r} = \frac{rh}{2(h - r)}$$
If $h \le 2r$, then $A'(x) \ne 0$ for all $x \le r$, so in this case the critical points are the endpoints, $x = 0, r$.
$A(0) = 0$; $A(r) = 2\pi r^2$
If $h \ge 2r$, then $0 \le \dfrac{rh}{2(h-r)} \le r$, so this is a critical point; $A\left(\dfrac{rh}{2(h-r)}\right) = \dfrac{\pi rh^2}{2(h-r)}$.
$A'(x)$ goes from positive to negative at $x = \dfrac{rh}{2(h-r)}$.

Maximum area at $x = \dfrac{rh}{2(h-r)}$ if $h \ge 2r$;
$x = r$ otherwise.

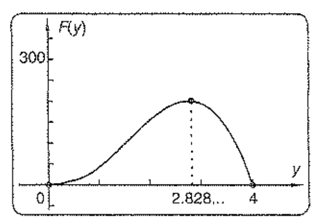
c. From part b, the maximal cylinder degenerates to two circular bases if the radius of the cone is at least half the height.

29. Maximize $V = \pi y^2 x$.
Ellipse equation is $(x/9)^2 + (y/4)^2 = 1$, from which $y^2 = (16/81)(81 - x^2)$.
$\therefore V = (16\pi/81)(81x - x^3)$
Domain: $0 \le x \le 9$



The graph shows a maximum V at $x \approx 5.2$.
$V' = (16\pi/81)(81 - 3x^2) = (16\pi/27)(27 - x^2)$
$V' = 0$ at $x = \pm\sqrt{27} = \pm 5.196\ldots$
$-5.196\ldots$ is out of the domain.
$V(0) = V(9) = 0$; $V(\sqrt{27}) = 32\pi\sqrt{3} = 174.1\ldots$
At $x = 5.196\ldots$, $y^2 = (16/81)(81 - 27) = 32/3 \Rightarrow y = \sqrt{32/3} = 3.2659\ldots$
$\therefore$ maximum volume ≈ 174.1 cm^3 at radius ≈ 3.27 m and height ≈ 5.20 m.

30. Maximize $C(y) = \pi y^2 x$, the area of the cylinder.
The parabola has an equation of the form $x = ay^2 + 16$.
$0 = a \cdot 16 + 16 \Rightarrow a = -1 \Rightarrow x = 16 - y^2$
$V(y) = \pi y^2(16 - y^2) = \pi(16y^2 - y^4)$
Domain: $0 \le y \le 4$



The graph shows a maximum $V(y)$ at $y \approx 2.8$.
$C'(y) = \pi(32y - 4y^3) = 4\pi y(8 - y^2) = 0$ at $y = 0, \pm\sqrt{8}$.
$y = -\sqrt{8}$ is out of the domain.
$C(0) = C(4) = 0$, $C(\sqrt{8}) = 64\pi = 201.0619\ldots$
Maximum $C(y)$ at $y = \sqrt{8}$.
At $y = \sqrt{8}$, $x = 8$.
Maximal cylinder has radius $= \sqrt{8} \approx 2.83$ m, height = 8 m, and volume $= 64\pi \approx 201.1$ m^3.
Maximize $F(y)$, the volume of the frustum.
Note that $V_f = (1/3)\pi h(R^2 + r^2 + Rr)$, where